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MAPPINGS OF VECTOR SPACES AND THE THEORY OF MATRICES

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1. Introduction. The concept of a matrix is best introduced, it is now generally agreed, by way of the theory of finite-dimensional vector spaces (sometimes called linear spaces). Such a space may, if one prefers, be defined abstractly [1; p. 167]. One may, however, define a vector space as a set of n-tuples (from some field). This entails no loss of generality, while having the advantage of greater familiarity. The operations in the space (addition and scalar multiplication) may then be defined precisely as if the elements of the n-tuple were just the rectangular components of a vector as ordinarily conceived, say, in physical problems (the sum of two vectors is obtained by adding their respective components, and so on). Again, one may think of the space as a set of single-rowed matrices, for here also the two operations are defined in exactly the same manner.

In any vector space there may exist subsets which are closed with respect to addition and scalar multiplication, a set being closed with respect to an operation if every operation on members of the set yields another member of the set. Such a set satisfies the postulates of a vector space, and is thus a subspace of the original space. In one approach to the theory of matrices, a matrix is regarded as simply a linear operator mapping the space either onto itself or onto a proper subspace of itself. By a linear operation one means an operation T such that $T(c_1V_1+c_2V_2)=c_1T(V_1)+c_2T(V_2)$ for any V_1, V_2 members of the space. This approach is, from many points of view, the most enlightening of all the various introductions to matrix theory. It is, for example, very natural from the point of view of geometry, where the behavior of a space under linear mappings is of fundamental importance. Such mappings, appearing as linear transformations, also have many important applications to physical problems. Again, the problem, in the theory of equations, of the solution of systems of linear equations appears naturally as an inverse mapping problem. This permits, for example, the number of independent solutions of the system to be the dimension of the subspace mapped by the coefficient matrix into the zero vector. This particular approach to the theory of matrices is found efficacious, moreover, in providing proofs which are both simple and elegant for many of the theorems of the algebra of matrices.

It is intended to illustrate this simplicity here by presenting a proof by vector space methods of a theorem of Frobenius on the ranks of matric products. For this presentation it is expected that the reader has some acquaintance with the fundamental concepts of vector spaces (independence, basis, and dimension). In addition, a few of the elementary theorems on vector spaces will be

used without proof (*e.g., Any independent subset of a space forms part of a basis,* and *A space has dimension greater than or equal to that of any of its subspaces*). The reader is also expected to know the definitions of the ordinary matrix operations (addition, multiplication, and scalar multiplication).

2. Linear mappings of vector spaces. The base space S will be chosen to be the set of all n-tuples from some field (the field of real numbers, if the reader prefers). This space of n-tuples is easily seen to satisfy the abstract postulates of a vector space, and in fact every finite-dimensional vector space is known to be isomorphic with some such space [1; p. 176]. Thus unless one prefers the abstract formulation, consideration may as well be limited to spaces of such n-tuples. As indicated above, a vector may be regarded as a matrix with a single row of n elements. This permits a definition of multiplication (on the right) of vectors by any matrix having n rows. A linear mapping of S (or a subspace of S) will then be defined to be the process of multiplying the vectors of the space (on the right) by a fixed n-by-n matrix A. If the reader prefers the abstract definition of the linear mapping, the matrix is simply the symbol of the operation, and the matrix, if one wishes to exhibit it at all, appears as the set of coefficients (in terms of a basis) of the images of the base vectors. The image of S under the mapping will be designated SA.

In proving the theorem of Frobenius, use is made of the two lemmas following, and since they may be new to some readers, their proofs are included. Let S_1 be a subspace of S, and let $N(A)$ be the set of all $V \in S$ for which $VA = 0$, then the intersection $N_1 = N \cap S_1$ is the set of all members of S_1 mapped into zero. If B is any space, let $d(B)$ denote the dimension of B.

LEMMA 1. *The sets N_1 and S_1A are subspaces such that $d(S_1) = d(N_1) + d(S_1A)$.*

It is easily verified that N_1 and S_1A are vector spaces. This may be checked directly, or one may cite the fact that N_1 is the intersection of subspaces N and S_1, while S_1A is the image of S_1 under a homomorphic mapping. The dimensions $d(N_1)$ and $d(S_1A)$ therefore exist. Let the set $\{V_i\}$ be a basis for N_1 and $\{W_j\}$ a basis for S_1A. Clearly there exists some set $\{U_j\}$ of inverse images (*i.e.*, such that $U_jA = W_j$). The set $\{V_i, U_j\}$ is independent, for any linear relation among its members would map into a linear relation among the members of the basis $\{W_j\}$. Thus $d(S_1A) + d(N_1) \leqq d(S_1)$.

But $VA \in S_1A$ for any $V \in S_1$, and so $VA = \sum c_jW_j$. Thus $V - \sum c_jU_j$ maps into zero, and being therefore in N_1, is expressible in terms of $\{V_i\}$. Hence V is expressible in terms of $\{V_i, U_j\}$ and so $d(S_1) - d(N_1) \leqq d(S_1A)$. This completes the proof.

For the matrix A of the mapping, the *rank* $\rho(A)$ and the so-called *nullity* $\nu(A)$ are defined as $\rho(A) = d(SA)$ and $\nu(A) = d(N(A))$. Since $d(S) = n$, the lemma implies the relation $\rho(A) + \nu(A) = n$. The rank so defined may be shown to be equivalent to other definitions of rank, such as the number of rows of the largest non-zero minor, or the number of independent rows or columns.

This lemma also yields interesting results in the theory of equations. Thus the system of equations $VA=Q$ will be solvable if and only if $Q\in SA$, and the number of independent parameters in the solution will be precisely $\nu(A)$. That is, if V maps into Q, then the solutions are formed by the residue class of V modulo $N(A)$.

Now let S_1 and S_2 be subspaces of S. The intersection $S_1\cap S_2$ and the sum S_1+S_2 (the set of all sums of vectors from S_1 and S_2) are known to be themselves vector spaces [1; p. 170, or by checking directly].

LEMMA 2. $d(S_1+S_2)=d(S_1)+d(S_2)-d(S_1\cap S_2)$.*

Since $S_1\cap S_2$ is a subspace of both S_1 and S_2, its basis is a part of a basis for either. If $\{U_i\}$ is a basis for $S_1\cap S_2$, then bases for S_1 and S_2 may be chosen respectively as $\{U_i, V_j\}$ and $\{U_i, W_k\}$. Clearly every vector in S_1+S_2 is expressible in terms of the set $\{U_i, V_j, W_k\}$, and this set contains $d(S_1)+d(S_2)-d(S_1\cap S_2)$ vectors. Further, this set is independent, for otherwise there is either a linear relation among some or all of the members of a basis ($\{U_i, V_j\}$ or $\{U_i, W_k\}$), or there exists some linear combination W of members of $\{W_k\}$ which is expressible in terms of $\{U_i, V_j\}$. But then $W\in S_1\cap S_2$ and so again there is a relation among the members of the basis $\{U_i, W_k\}$. This completes the proof.

The above lemmas make possible an easy proof of the theorem of Frobenius. Let A, B, and C be n-by-n matrices; then

THEOREM 1. $\nu(B)+\nu(ABC)\leqq\nu(AB)+\nu(BC)$.†

Consider the mapping: S onto SA. If $V\in N(AB)$, then clearly the mapped element VA satisfies $(VA)B=0$. Thus since $N(AB)A$ is the image of $N(AB)$, it follows that $N(AB)A=SA\cap N(B)$. Then by Lemma 1, $\nu(AB)=\nu(A)+d(SA\cap N(B))$, and hence by Lemma 2, $\nu(AB)=\nu(A)+d(SA)+\nu(B)-d(SA+N(B))$. But it was noted above that $\nu(A)+d(SA)=n$, and thus

$$\nu(AB)-\nu(B)=n-d(SA+N(B)). \tag{1}$$

Substituting BC for B gives

$$\nu(ABC)-\nu(BC)=n-d(SA+N(BC)). \tag{2}$$

But clearly every solution of $VB=0$ is a solution of $VBC=0$, so that $N(B)$ is a subspace of $N(BC)$. Thus $SA+N(B)$ cannot have a dimension greater than that of $SA+N(BC)$. Relations (1) and (2) thus yield $\nu(ABC)-\nu(BC)\leqq\nu(AB)-\nu(B)$, which is the desired result.

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* See [1; p. 179].

† See [2; p. 11]. The essential ideas of the proofs here given are to be found in that of MacDuffee.

From this theorem follow, as corollaries, two very useful formulas (Sylvester's law of nullity):

COROLLARY 1. $\nu(AB) \geqq \max(\nu(A), \nu(B))$.

Proof: In Theorem 1, first let $A=0$, and then let $C=0$.

COROLLARY 2. $\nu(AB) \leqq \nu(A)+\nu(B)$.

Proof: In theorem 1, let $B=I$.

References

1. Garrett Birkhoff and Saunders MacLane, A Survey of Modern Algebra, The Macmillan Co., New York, 1948.

2. C. C. MacDuffee, The Theory of Matrices, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 2, Springer, Berlin, 1933.

CONTINUOUS FUNCTIONS WITHOUT DERIVATIVES

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In this note we give an elementary example of a class of continuous functions which fail to possess a derivative. The known examples of this phenomenon all involve limiting processes in one form or another and are therefore somewhat complicated.

We shall restrict the discussion to the unit interval so that any number may be written in the form

$$x = .x_1x_2x_3\cdots$$

where each x_i is one of the numbers 0, 1, 2, $\cdots$, $b-1$ where b is an integer used as the base. Once chosen, b is to be kept fixed. We further demand $b>2$.

We say a run begins on x_i if $x_{i-1}\neq x_i$. Conventionally we agree that a run always begins on x_1. We now form a function $f[x]$ by setting

$$f[x] = .u_1u_2u_3\cdots$$

where u_i is the number of runs reduced modulo 2 that have occurred in the first i places of the b-adic expansion of x. The functional values are therefore represented in base 2.

THEOREM 1. *$f[x]$ is a continuous function.*

Proof. Suppose $2^{-p}<\epsilon$ where ϵ has been fixed in advance. Given any point

$$x^0 = .x_1^0x_2^0x_3^0\cdots,$$

we choose a neighboring point x such that

$$|x - x^0| < b^{-p}. \tag{1}$$

This can occur in two ways. We may have

$$x_i = \overset{0}{x_i}, \qquad i = 1, 2, \cdots, p.$$

[In this case we suppose that the remaining places in the expansion of the difference (1) are not all filled with the value $b-1$.] Since each u_i depends only on its predecessors, we have

$$u_i = \overset{0}{u_i}, \qquad i = 1, 2, \cdots, m \text{ where } m \geqq p$$

so that

$$|f[x] - f[x^0]| < \epsilon. \tag{2}$$

We also have the possibility

$$\begin{aligned} x_i &= \overset{0}{x_i}, && i = 1, 2, \cdots, k-1 \\ x_k &= \overset{0}{x_k} + 1 \\ x_i &= 0, \quad \overset{0}{x_i} = b - 1, && i = k+1, k+2, \cdots, p. \end{aligned}$$

[In certain cases it may be necessary to have the last of these relationships hold for $i=p+1$. A third possibility, $x_k=x_k^0-1$, is in all respects the dual of the case we discuss.]

If in particular $k=1$, we must have

$$\begin{aligned} u_1 &= \overset{0}{u_1} = 1 \\ u_2 &= \overset{0}{u_2} = 0, \end{aligned}$$

since a run must begin on x_2 and on x_2^0,

$$u_i = \overset{0}{u_i} = 0, \qquad 2 < i \leqq p$$

since no run begins in the next p places. Consequently

$$|f[x] - f[x^0]| < \epsilon.$$

For the general case we use three initial dots to indicate that the symbols in the first $k-1$ places agree. Suppose we have

$$\left.\begin{matrix} \cdots \overset{0}{x_k} + 1 & 0 & 0 & \cdots \\ \cdots \overset{0}{x_k} & b-1 & b-1 & \cdots \end{matrix}\right\} \text{ repetitions holding to the } p\text{th place,}$$

for the respective expansions of x and x^0. If x_{k-1}^0 differs from x_k^0 and from x_k^0+1, runs begin in the kth place for both x and x^0 so that (2) holds. But if x_{k-1}^0 agrees with one of these two values, a run will begin on the kth place of one expansion

and not on the other so that we can represent the two functional values by

$$\left.\begin{matrix}\cdots 1\ 0\ 0\cdots\\ \cdots 0\ 1\ 1\cdots\end{matrix}\right\}\ \text{repetitions holding to the } p\text{th place,}$$

where one of these values will be associated with x^0, although this is not necessarily the first value, and the other with x. These values satisfy (2), completing the proof.

THEOREM 2. *$f[x]$ has no derivative.*

Proof. Let us set $x=x_i^0$ except when $i=p$. Then

$$|x - x^0| = kb^{-p}, \qquad 0 < k \leqq b-1.$$

We secure five different cases tabulated below which depend upon the relationships existing between x_{p-1}^0, x_p^0, and x_{p+1}^0. These entries appear in the second line of each pair. At the right we indicate the corresponding entries in the dyadic system of the corresponding function values on the assumption that $u_{p-1}=0$. If $u_{p-1}=1$, then the zeros and ones are to be interchanged, but this will not affect the essential argument in any way.

Case I.

	u_p	u_{p+1}
$x_{p-1}^0 \neq x_p \neq x_{p+1}^0$	$\cdots 1$	$0\cdots$
$x_{p-1}^0 = x_p = x_{p+1}^0$	$\cdots 0$	$0\cdots$

Case II.

$x_{p-1}^0 \neq x_p = x_{p+1}^0$	$\cdots 1$	$1\cdots$
$x_{p-1}^0 = x_p \neq x_{p+1}^0$	$\cdots 0$	$1\cdots$

Case III.

$x_{p-1}^0 = x_p \neq x_{p+1}^0$	$\cdots 0$	$1\cdots$
$x_{p-1}^0 \neq x_p = x_{p+1}^0$	$\cdots 1$	$1\cdots$

Case IV.

$x_{p-1}^0 = x_{p+1}^0$		
$x_{p-1}^0 = x_p = x_{p+1}^0$	$\cdots 0$	$0\cdots$
$x_{p-1}^0 \neq x_p \neq x_{p+1}^0$	$\cdots 1$	$0\cdots$

Case IVa.

$x_{p-1}^0 \neq x_{p+1}^0$		
$x_{p-1}^0 = x_p \neq x_{p+1}^0$	$\cdots 0$	1
$x_{p-1}^0 \neq x_p \neq x_{p+1}^0$	$\cdots 1$	0

In the first four cases, the function values differ only in the pth place. We therefore have

$$|f[x] - f[x^0]| = 2^{-p}.$$

For these cases we therefore have

$$\frac{2^{-p}}{kb^{-p}} = \left|\frac{f[x] - f[x^0]}{x - x^0}\right|.$$

If x^0 has the property that an infinite number of values of p can be found such that one of the four cases first listed holds, then we can always choose a neighboring point x such that the differential quotient can be made arbitrarily large in absolute value by choosing p sufficiently large. If Case IVa holds when $i \geqq p_0$, then a run must begin on each digit x, $i \geqq p_0$ so that for $p \geqq p_0$ we have

$$f[x^0] = \cdots 1\ 0\ 1\ 0 \cdots$$
$$f[x] = \cdots 0\ 1\ 0\ 1 \cdots$$

in accordance with our choice of x. Hence the absolute value of their difference is easily computed to be $[4/3]2^{-[p+1]}$. Again the absolute value of the differential quotient can be made arbitrarily large by choosing p sufficiently large so that no derivative can exist.

If $b=2$, the function reduces to a straight line, but the non-trivial cases possess an interesting intrinsic property embodied in the following theorem.

THEOREM 3. *If $1/2<t<1$ and if t is not representable as a finite sum of elements of the form 2^{-n}, then $f[x]-t=0$ has a non-denumerable number of solutions.*

Proof. Consider the decimal expansion of t in base 2. The restriction on t implies that t contains an infinite number of runs. The expansion therefore contains an infinite number of 1's preceded by 0. It is possible to generate 1 in the pth place preceded by 0 by $b-1$ different choices of x_p. Hence there are $[b-1]^a$ such numbers where a is the cardinal number of the integers. This theorem throws considerable light on the non-existence of the derivative.

SUMS OF POWERS OF NUMBERS HAVING A GIVEN EXPONENT MODULO A PRIME

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In his *Disquisitiones Arithmeticae*, C. F. Gauss [3] showed that the sum of the primitive roots of a prime p is congruent modulo p to $(-1)^k$ if $p-1$ is a product of k distinct primes, and to zero if $p-1$ contains a square factor; or, as we may now state it more concisely, to $\mu(p-1)$. This theorem was later generalized by Stern [5], who proved that the sum of the numbers belonging to any divisor d of $p-1$,where p is a prime, is congruent modulo p to $\mu(d)$. A further generalization by Forsyth [2] established congruence theorems for the sum of the nth powers of the primitive roots of a prime, but the statement of his results was so complicated by the enumeration of various special cases that Dickson, in his *History* [1], summarized only a small part of them and dismissed the rest as "not so simple." The purpose of this paper is to obtain one simple congruence relationship which will express the sum of the nth powers of the numbers belonging to any divisor d of $p-1$ in terms of the Möbius function and the Euler ϕ-function.

If g is one number belonging to d modulo p, then the set of all numbers belonging to d is

$$g^j \qquad [0 < j \leqq d, (j, d) = 1]. \tag{1}$$

When the members of set (1) are raised to the power n, each member of the new set will be a number belonging to d/d', where $d' = (n, d)$. However, since there are only $\phi(d/d')$ numbers belonging to d/d', the $\phi(d)$ members of the new set will not be distinct modulo p, except in the trivial case where $d' = 1$ and the new set is identical with (1). The members of the new set are all of the numbers

$$g^{nj} \qquad [0 < j \leqq d, (j, d) = 1]. \tag{2}$$

A number g^{nx}, with no restriction on x, will be congruent modulo p to any given member g^{nj_1} of set (2) if and only if

$$nx \equiv nj_1 \pmod{d},$$

which in turn is true if and only if

$$x \equiv j_1 \pmod{d/d'}.$$

There is exactly one solution for x in the range $0 < x \leqq d/d'$, and d' solutions in the range $0 < x \leqq d$; however, not all of these values of x are possible values of j [corresponding to members of (2)], because of the restriction $(j, d) = 1$. To determine how many of these solutions are values of j, we must determine how many members of the set

$$j_1 + dt/d' \qquad (t = 0, 1, 2, \cdots, d' - 1), \tag{3}$$

consisting of all solutions of the congruence, are prime to d. Since all of the members of (3) are prime to d/d', any member will be prime to d if and only if it is prime to d'.

Let $d' = \prod p_i^{\alpha_i} \prod q_i^{\beta_i}$, where the p_i and q_i are primes such that $(p_i, d/d') = 1$ and $(q_i, d/d') = q_i$. The members of set (3) which are prime to d' are those which are not divisible by any p_i, since none of them is divisible by any q_i. For the congruence $j_1 + dt/d' \equiv 0 \pmod{q_i}$ has no solution inasmuch as $(q_i, d/d') = q_i$ does not divide j_1, which is prime to d and *a fortiori* to d/d'. Now corresponding to each p_i there is one t in the range $0 \leqq t < p_i$ such that $j_1 + dt/d' \equiv 0 \pmod{p_i}$, and d'/p_i such values in the range $0 \leqq t < d'$. The values of t for which the above congruence is satisfied modulo p_i and those for which it is satisfied modulo p_j are not necessarily distinct. However, for each duplication the congruence must be satisfied modulo $p_i p_j$, and so the number of such duplications is one in the range $0 \leqq t < p_i p_j$ and $d'/p_i p_j$ in the range $0 \leqq t < d'$. In like manner, for each t which satisfies the congruence modulo p_i, modulo p_j, $\cdots$, and modulo p_s, the congruence must be satisfied modulo $p_i p_j \cdots p_s$, so that there is just one such t in the range $0 \leqq t < p_i p_j \cdots p_s$ and $d'/p_i p_j \cdots p_s$ such values of t in the range $0 \leqq t < d'$.

Consequently, the number N of members of set (3) which are prime to d', being the number of values of $j_1 + dt/d'$ $(t = 1, 2, \cdots, d' - 1)$ possessing none of the properties of divisibility by a p_i, is equal (by a well-known combinatorial theorem [4]) to

$$d' - \sum d'/p_i + \sum d'/p_{i_1} p_{i_2} - \cdots + (-1)^k \sum d'/p_{i_1} p_{i_2} \cdots p_{i_k},$$

where the summations are taken over all combinations of prime divisors of d' such that $(p_i, d/d') = 1$. This may be written

$$N = d' \prod_{p_i} (1 - 1/p_i).$$

If we now denote by r_i those prime factors of d which do not divide d', then the p_i, q_i, and r_i are all of the prime divisors of d, and the q_i and r_i are all of the prime divisors of d/d'. Thus $\phi(d) = d \prod (1 - 1/p_i)(1 - 1/q_i)(1 - 1/r_i)$ and $\phi(d/d') = (d/d') \prod (1 - 1/q_i)(1 - 1/r_i)$, so that

$$\phi(d)/\phi(d/d') = d' \prod (1 - 1/p_i) = N.$$

Since the expression for N is independent of j_1, we see that all of the members of (2) fall into subsets each containing the same number of members congruent to each other modulo p; or, in other words, any number which does appear in (2) is repeated exactly $\phi(d)/\phi(d/d')$ times. Now since the $\phi(d)$ numbers in (2) belong to d/d', only $\phi(d/d')$ of them can be distinct, so that in order to account for the total of $\phi(d)$ numbers, every number belonging to d/d' must appear. Therefore the set of numbers obtained by raising to the nth power all of the numbers belonging to d modulo p consists of all of the numbers belonging to d/d' modulo p, each repeated $\phi(d)/\phi(d/d')$ times. But since the sum of

the numbers belonging to d/d' is congruent modulo p to $\mu(d/d')$, the sum of the nth powers of the numbers belonging to d must be congruent modulo p to

$$\frac{\phi(d)}{\phi(d/d')}\mu(d/d').$$

We have thus proved the following theorem:

THEOREM I: *If the numbers g_d are all of the numbers belonging to d modulo p, p being any prime and d any divisor of $p-1$, then for any n*

$$\sum g_d^n \equiv \frac{\phi(d)}{\phi[d/(n, d)]}\mu[d/(n, d)] \pmod{p},$$

where ϕ is the Euler ϕ-function and μ is the Möbius function.

We shall now consider the function

$$F(x, n) = \sum_{d|x} \frac{\phi(d)}{\phi[d/(n, d)]}\mu[d/(n, d)]$$

which by Theorem I is congruent modulo p to the sum of the nth powers of all numbers belonging modulo p to any of the divisors of x, where p is any prime such that x divides $p-1$. (There are of course infinitely many such primes for any given x.) If $(d, d')=1$, we have

$$\left\{\frac{\phi(d)}{\phi[d/(n, d)]}\mu[d/(n, d)]\right\}\left\{\frac{\phi(d')}{\phi[d'/(n, d')]}\mu[d'/(n, d')]\right\}$$

$$= \frac{\phi(dd')}{\phi[dd'/(n, d)(n, d')]}\mu[dd'/(n, d)(n, d')] = \frac{\phi(dd')}{\phi[dd'/(n, dd')]}\mu[dd'/(n, dd')]$$

so that $F(x, n)$ has the multiplicative property [4]

$$F(x, n)F(x', n) = F(xx', n) \quad \text{when} \quad (x, x') = 1.$$

We proceed to evaluate $F(q^\alpha, n)$, where q is a prime. If $(n, q^\alpha)=q^\beta(0\leqq\beta<\alpha)$, then

$$F(q^\alpha, n) = \sum_{d|q^\beta}\{\phi(d)\} + \frac{\phi(q^{\beta+1})}{\phi(q)}\mu(q) = q^\beta - q^\beta = 0.$$

If $(n, q^\alpha)=q^\alpha$,

$$F(q^\alpha, n) = \sum_{d|q^\alpha}\phi(d) = q^\alpha.$$

Using the multiplicative property of $F(x, n)$, we conclude that if n is not a multiple of x, $F(x, n)=0$, for there would be at least one $q_i^{\alpha_i}$ dividing x such that $(n, q_i^{\alpha_i})=q_i^{\beta_i}$, $(0\leqq\beta_i<\alpha_i)$, and consequently at least one zero factor in

$\prod_i F(q_i^{\alpha_i}, n) = F(x, n)$. On the other hand, if n is a multiple of x, we see that $F(x, n) = \prod_i q_i^{\alpha_i} = x$. Thus we have proved

THEOREM II: *The sum of the nth powers of all numbers belonging modulo p to any of the divisors of x, where p is any prime such that x divides $p-1$, is congruent modulo p to x or to zero, according as n is or is not a multiple of x.*

As a special case, if we take $x = p-1$, we have the well-known theorem that the sum of the nth powers of all the numbers from 1 to $p-1$ inclusive is congruent modulo p to $p-1$ or to zero according as n is or is not a multiple of $p-1$; for every number from 1 to $p-1$ belongs modulo p to one of the divisors of $p-1$.

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4. Hardy, G. H., and Wright, E. M., An Introduction to the Theory of Numbers, Oxford, 1938, pp. 232, 235.
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Additional Remarks by H. S. Zuckerman, University of Washington. It is interesting to note that it is possible to prove Theorem II first and then obtain Theorem I from it. We let $f(d, n) = \sum g_d^n$ and have $F(x, n) = \sum_{d|x} f(d, n)$ and $x \mid (p-1)$. Now $u^x \equiv 1 \pmod p$ if and only if u_0 belongs to the exponent d modulo p and $d \mid x$ so we have $F(x, n) \equiv \sum u^n \pmod p$ where the sum is taken over all u (mod p) such that $u^x \equiv 1 \pmod p$. If u_0 belongs to the exponent x modulo p then $(u_0 u)^x \equiv 1 \pmod p$, $(u_0, p) = 1$ and hence the set of $u_0 u$ is congruent to the set of u modulo p, in some order. Then $\sum u^n \equiv \sum (u_0 u)^n \equiv u_0^n \sum u^n \pmod p$ and $\sum u^n \equiv 0 \pmod p$ unless $u_0^n \equiv 1 \pmod p$, that is $x \mid n$. We then find

$$\text{(A)} \qquad F(x, n) \equiv \begin{cases} 0 \pmod p & \text{if } x \nmid n, \\ x \pmod p & \text{if } x \mid n, \end{cases}$$

and this is Theorem II.

From the Möbius inversion formula we find $f(x, n) = \sum_{d|x} F(d, n)\mu(x/d)$ and, using (A), $f(x, n) \equiv \sum d\mu(x/d) \pmod p$ where the sum extends over all d that divide both x and n. We can write $x/(x, n) = a$, $x = aa'b$, where $(aa', b) = 1$ and every prime dividing a' also divides a. Then $(x, n) = a'b$ and we have

$$\sum_{d|(x,n)} d\mu\left(\frac{x}{d}\right) = \sum_{d|a'b} d\mu\left(\frac{aa'b}{d}\right) = \sum_{\delta|a'b} \frac{a'b}{\delta}\mu(a\delta)$$

$$= \sum_{\delta|b} \frac{a'b}{\delta}\mu(a)\mu(\delta) = a'\mu(a)\phi(b)$$

since $\mu(a\delta) = 0$ unless $(a, \delta) = 1$. In order to express this in terms of x and n we

insert a factor $\phi(a)$ and find

$$a'\mu(a)\phi(b) = \frac{a'\phi(a)\phi(b)}{\phi(a)}\mu(a) = \frac{\phi(aa')\phi(b)}{\phi(a)}\mu(a) = \frac{\phi(aa'b)}{\phi(a)}\mu(a)$$
$$= \frac{\phi(x)}{\phi\left(\frac{x}{(x, n)}\right)}\mu\left(\frac{x}{(x, n)}\right).$$

Now we have

$$f(x, n) \equiv \frac{\phi(x)}{\phi\left(\frac{x}{(x, n)}\right)}\mu\left(\frac{x}{(x, n)}\right) \pmod{p}$$

which is Theorem I.

MAXIMA AND MINIMA UNDER RESTRAINT*

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1. Introduction. The problem of finding the maximum or minimum of a function whose several variables are required to satisfy one or more auxiliary equations is treated extensively in the literature and in standard texts. In general, the working rules are given without an adequate explanation of why they work. The methods of solution usually fall into one of three categories.

First, the auxiliary equations may be used to eliminate an equal number of dependent variables from the given function. In attempting to use this method, the elimination may prove to be difficult or even impossible in finite form. However, since the method reduces the problem to one without restraint, it need not concern us further.

Second, the differentials of all functions may be taken. If the differentials of the dependent variables are then eliminated, enough equations are formed to solve for all the quantities involved. This is the method used here. It has many variations.

Third, Lagrange multipliers may be used. The rules for their use are essentially short-cuts in the solution by the second method above. In addition, the use of these multipliers formalizes the problem and lends a kind of symmetry to the solution which the other methods do not possess.

* Adapted from an address as retiring Chairman of the Southeastern Section of the Mathematical Association of America delivered at Vanderbilt University, March 16, 1951.

The purpose of the present article is (1) to effect an algebraic solution by the second method, (2) to show its connection with the use of Lagrange multipliers, and (3) to give an analytic and geometric interpretation often overlooked. No special cases are considered.

All functions involved are assumed to be continuous in a given region R and to have continuous first and second partial derivatives with respect to each of the variables. Additional restrictions upon these functions are stated in the appropriate places.

2. The general problem. Let $f(x_1, x_2, \cdots, x_n)$ be a function defined in the region R and whose n arguments are restrained to satisfy the m auxiliary equations

$$g^k(x_1, x_2, \cdots, x_n) = a_k, \qquad k = 1, 2, \cdots, m < n. \tag{2.1}$$

If subscripts on the functions are used to indicate partial differentiation with respect to the correspondingly numbered variable, it is assumed that the matrix

$$(g_i^k), \qquad k = 1, 2, \cdots, m; \; i = 1, 2, \cdots, n,$$

is non-singular in R.

A *stationary point* of the function f is defined as a point at which its differential is zero; *i.e.*,

$$df = \sum_{i=1}^{n} f_i dx_i = 0, \tag{2.2}$$

where the differentials of the x_i are subject to the m equations

$$dg^k = \sum_{i=1}^{n} g_i^k dx_i = 0, \qquad k = 1, 2, \cdots, m. \tag{2.3}$$

Thus the stationary points of f may include, in addition to the points at which it attains its maxima and minima, some points at which it attains neither.

Since the matrix above is non-singular, it is possible to number the variables so that the determinant $|g_1^1, g_2^2, \cdots, g_m^m|$ does not vanish, then to take the variables $x_1, x_2, \cdots, x_m$ as dependent, and to eliminate the differentials of these variables from (2.2) by a choice of multipliers $\lambda_1, \cdots, \lambda_m$ such that

$$f_j - \sum_{k=1}^{m} \lambda_k g_j^k = 0, \qquad j = 1, 2, \cdots, m. \tag{2.4}$$

With this choice the equation in (2.2) is transformed into

$$\sum_{i=m+1}^{n} \left(f_i - \sum_{k=1}^{m} \lambda_k g_i^k \right) dx_i = 0.$$

Since this is an identity in the differentials appearing here, the coefficient of each differential is zero. Hence it is possible to extend (2.4) to all variables and write

$$f_i - \sum_{k=1}^{m} \lambda_k g_i^k = 0, \qquad i = 1, 2, \cdots, n. \tag{2.5}$$

These n equations together with the m equations in (2.1) serve to determine the coordinates of the stationary points and the values of all the multipliers. Since any other choice of dependent variables would lead to the same set of equations as in (2.5), the set of multipliers at each stationary point is unique but will in general be different for different stationary points.

3. The second order conditions. If the stationary point is a minimum, it is necessary that the second differential of f be positive or zero there; it is sufficient to insure a minimum if the differential is positive. The only tests which are convenient to apply are those based on this sufficiency.

The second differential of f is given by

$$d^2f = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} dx_i dx_j + \sum_{k=1}^{m} f_k d^2x_k > 0. \tag{3.1}$$

The second differentials of the g-functions are similar except that they must vanish; *i.e.*,

$$d^2g^k = 0, \qquad k = 1, 2, \cdots, m. \tag{3.2}$$

The left member of (3.1) can be made into a quadratic form in the differentials of all the variables by adding appropriate multiples of the second differentials of the g-functions. Thus

$$d^2f - \sum_{k=1}^{m} \lambda_k d^2g^k = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(f_{ij} - \sum_{k=1}^{m} \lambda_k g_{ij}^k \right) dx_i dx_j > 0, \tag{3.3}$$

where the λ_k are the same multipliers as those defined by the equations in (2.4). The conditions under which the left side of (3.3) is a positive definite quadratic form subject to the m linear restraints in (2.3) are easily written down.*

Let

$$b_{ij} = f_{ij} - \sum_{k=1}^{m} \lambda_k g_{ij}^k, \qquad i, j = 1, 2, \cdots, n;$$

* See H. B. Mann, Quadratic forms with linear restraints, this MONTHLY, vol. 50, pp. 430–433. His conditions for a negative definite form under restraint should read $(-1)^i h_i > 0$ instead of $(-1)^{p+i} h_i > 0$.

$$h_t = \begin{vmatrix} 0 & \cdots & 0 & g_1^1 & \cdots & g_t^1 \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ 0 & \cdots & 0 & g_1^m & \cdots & g_t^m \\ g_1^1 & \cdots & g_1^m & b_{11} & \cdots & b_{1t} \\ \cdot & & \cdot & \cdot & & \cdot \\ g_t^1 & \cdots & g_t^m & b_{t1} & \cdots & b_{tt} \end{vmatrix}, \qquad t = m, \cdots, n.$$

The expression in (3.3) will be positive definite if and only if h_t has the sign of $(-1)^m$ for all the specified values of t. These conditions for a minimum are only sufficient.

If the stationary point is a maximum, the sense of the inequality in (3.3) is reversed. The conditions upon h_t for this case are easily derived from the conditions for a positive definite form.

This negative definite form can be changed into one which is positive definite by multiplying by -1. At the same time let each equation in (2.3) also be multiplied by -1. Let k_t be the determinant associated with this form and corresponding to h_t of the previous form. Since k_t and h_t are determinants of order $m+t$, we have that

$$h_t = (-1)^{m+t}k_t = (-1)^{2m+t}\left| k_t \right| = (-1)^t \left| k_t \right|.$$

Therefore, for a negative definite quadratic form under restraint, h_t has the sign of $(-1)^t$ for the values of t indicated.

If the signs of h_t do not follow either pattern, the point is neither a maximum nor a minimum. If, however, some of the h_t are zero but otherwise conform to the pattern of signs, the stationary point cannot be classified by this test.

4. An interpretation. Since every set of equations to be solved contains the equations of (2.1), the a-constants of these equations are parameters of the solution. Hence, the coordinates of each stationary point, the value of the multipliers there, and the stationary value of f are all functions of these parameters.

Let c be the stationary value of f, then

$$c = f(x_1, x_2, \cdots, x_n) = F(a_1, a_2, \cdots, a_m). \tag{4.1}$$

If now each equation in (2.1) is differentiated implicitly with respect to any one a_j, and if each derived equation is multiplied by the corresponding λ_k and collectively subtracted from the equation similarly derived from (4.1), the result rearranges to read

$$\frac{\partial c}{\partial a_j} = \lambda_j, \qquad j = 1, 2, \cdots, m. \tag{4.2}$$

The results in (4.2) furnish an analytic interpretation of the multipliers used in eliminating the dependent differentials from (2.2). Mathematicians have effectively prevented themselves from reaching this interpretation by always setting the g-functions equal to zero. Mathematical economists have of necessity fared better although, under the influence of the mathematicians, they have always transposed their constants to the left side of the equation before applying this method.

This interpretation has given the economists two important concepts. If f is the utility function of an individual, and if the first equation of (2.1) is the budget equation of the individual in which a_1 is his income, then λ_1 is the marginal utility of income. Again, if f is the cost of producing a certain amount of some good, and if $g^1 = a_1$ is the production equation, then the marginal cost of production in producing a_1 units of this good is λ_1. For a more detailed discussion of these concepts and a fuller explanation of the terminology the reader is referred to the last two items of the bibliography.

5. A second interpretation. Define the function

$$u = f(x_1, \cdots, x_n) - \sum_{k=1}^{m} \lambda_k g^k(x_1, \cdots, x_n) \tag{5.1}$$

as a point function in n-dimensional space. Consider this function at a stationary point of f as determined above with the λ_k taken as the multipliers associated with this point. If ds is the differential of arc along any arbitrary curve through this point, it follows from (2.5) that

$$\frac{du}{ds} = \sum_{i=1}^{n} \left(f_i - \sum_{k=1}^{m} \lambda_k g_i^k \right) \frac{dx_i}{ds} = 0; \tag{5.2}$$

i.e., the derivative of u is zero in every direction from this point.

In turn, the result in (5.2) implies that, at a stationary point of f, its directional derivative is a linear combination of the directional derivatives of the g-functions taken in the same direction; also, the coefficients of this linear combination are the multipliers associated with this point. This interpretation is interesting when visualized in three-space (using vector analysis) with either one or two restraining equations.

6. Lagrange multipliers. Lagrange first proposed the use of the function u about 1797.* From this fact these multipliers take their name. Like most others since, he gave no explanation beyond the statement that the rules he gave worked.

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* See the work cited here or the various reprints of his *Theorie des fonctions analytiques.*

These rules may now be formulated as a direct consequence of the facts indicated in (5.2). They are:

(1) form the function u as defined in (5.1) above;
(2) treat the m multipliers *as if* they were constant;
(3) treat the n variables *as if* they were independent;
(4) set the n derivatives of u equal to zero *as if* the function were unrestricted.

By following these rules, one is lead immediately to the equations of (2.5) which together with those in (2.1) are sufficient in number to determine all unknowns. In addition, the second differential of u yields the quadratic form in (3.3).

The function in (5.1) may be written in a slightly more general form as

$$v = f - \sum_{k=1}^{m} \lambda_k(g^k - a_k). \tag{6.1}$$

This function may then be treated, up to a certain point, *as if* the n x_i and the m λ_k were all independent variables. The first derivatives of v set equal to zero yield all the necessary equations. Also, one may form the determinants h_t from the second derivatives *as if* the function were without restriction. With this, the analogy ceases, for the determinants of lower orders are not relevant, and the signs of the others, for sufficiency, must be those given in section 3.

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MATHEMATICAL NOTES

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A CONTINUED FRACTION CIPHER

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The elementary properties of continued fractions form the basis of a cipher with unusual characteristics, the most interesting being that a message can be enciphered in blocks containing an arbitrary number of letters, different for each block, yet every such letter-group can be represented uniquely in cipher by a single pair of integers.

Let each letter of the alphabet be assigned a unique positive integer, $\alpha_i \neq 1$, $(i = 1, 2, \cdots, 26)$. A word ρ, or any consecutive letter sequence, after being expressed numerically as a series of integers, α_i, spaced (or separated by dashes), may be enciphered in blocks of any number Δ of letters, $(\Delta \neq 1)$, as follows.

Let the α_i's be consecutive partial quotients of the continued fraction

$$\beta = \alpha_a + \frac{1}{\alpha_b} + \frac{1}{\alpha_c} + \cdots .$$

The expression of β as an ordinary fraction gives the unique equivalent: $\beta = \phi/\delta$. Then ρ is written in cipher as the integer-pair $\phi - \delta$, or simply ϕ (space) δ.

Obviously, to decipher the sequence $\phi_1\ \delta_1\ \phi_2\ \delta_2 \cdots \phi_r\ \delta_r$, each ϕ_k/δ_k is expanded into a continued fraction; the partial quotients will be the α_i's of the "clear" letter-groups.

Especially interesting is the fact that a message of any length can be enciphered as a single (presumably large) pair of integers.

As an illustration of the method, consider the cipher sequence

566 43 4038 191 181 12 1289 91 289 129.

The quotient 566/43, expanded into a continued fraction, gives the partial quotients, 13, 6, 7. Assuming the simple choice of α_i's given by A = 2, B = 3, etc., these quotients represent the letters L E F. Similarly, 4038/191, with partial quotients 21, 7, 13, 2, gives the four letters, T F L A. Continuing, we easily obtain the remaining letter blocks N K, M E N, A C E D, all of which reads LEFT FLANK MENACED.

ON ASCOLI'S THEOREM

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Preliminary to proving Ascoli's Theorem in real variables, one proves the following theorem:

If $\{f_n(x)\}$ is an equicontinuous family for x in a fixed compact set C in n-space such that $\lim_{n\to\infty} f_n(x)$ exists on a set dense in C, then $\lim_{n\to\infty} f_n(x)$ exists for all x in C.

It is then quite natural to inquire if there exists a family of continuous functions which converges on a dense set and also diverges on a dense subset of C. It is the purpose of this paper to present a very elementary example of *a sequence of functions $\{f_n(x)\}$ (which are continuous and uniformly bounded on the interval $0 \leqq x \leqq 1$) which converges on a countable dense subset of $0 \leqq x \leqq 1$ and diverges elsewhere.*

For each fixed non-negative integer n, let

$$f_n(x) = \begin{cases} 2^{n+1}\left(x - \dfrac{j}{2^n}\right) & \text{for } \dfrac{j}{2^n} \leqq x \leqq \dfrac{2j+1}{2^{n+1}}, \\ -2^{n+1}\left(x - \dfrac{j+1}{2^n}\right) & \text{for } \dfrac{2j+1}{2^{n+1}} \leqq x \leqq \dfrac{j+1}{2^n}, \end{cases}$$

where $j = 0, 1, \cdots, 2^n - 1$. That is, $f_n(x)$ is that function whose graph is the polygon obtained by joining successively with straight lines the points in the plane

$$(0, 0), \left(\frac{1}{2^{n+1}}, 1\right), \left(\frac{1}{2^n}, 0\right), \cdots, \left(\frac{j}{2^n}, 0\right), \left(\frac{2j+1}{2^{n+1}}, 1\right), \left(\frac{j+1}{2^n}, 0\right), \cdots, (1, 0).$$

It is then evident, from the graphs of $f_{n+1}(x)$ and $f_n(x)$, that

$$f_{n+1}(x) = 2f_n(x)$$

for x in the leftmost and rightmost quarters of the interval

$$E_x\left[\frac{j}{2^n} \leqq x \leqq \frac{j+1}{2^n}\right],$$

while

$$f_{n+1}(x) = 2(1 - f_n(x))$$

in the two middle quarters of this interval. Thus, for any x in the unit interval and each positive integer n, either

$$f_{n+1}(x) = 2f_n(x) \quad \text{or} \quad f_{n+1}(x) = 2(1 - f_n(x)).$$

In other words, either

$$f_{n+1}(x) - f_n(x) = f_n(x) \quad \text{or} \quad f_{n+1}(x) - f_n(x) = 2 - 3f_n(x).$$

If $\lim_{n\to\infty} f_n(x)$ exists, then clearly

$$\lim_{n\to\infty} [f_{n+1}(x) - f_n(x)] = 0.$$

Thus, either there is only a finite number of n's such that

$$f_{n+1}(x) - f_n(x) = f_n(x),$$

or there is only a finite number of n's such that

$$f_{n+1}(x) - f_n(x) = 2 - 3f_n(x);$$

for otherwise we should have the absurdity

$$0 = \lim_{n\to\infty} f_n(x) = 2 - 3 \lim_{n\to\infty} f_n(x) = 2.$$

Hence, either there exists an integer N such that $n \geqq N$ implies $f_{n+1}(x) = 2(1-f_n(x))$, or there is an integer N such that $n \geqq N$ implies $f_{n+1}(x) = 2f_n(x)$. In the former case we see that $n \geqq N$ implies

$$f_{n+1}(x) - \tfrac{2}{3} = -2(f_n(x) - \tfrac{2}{3}),$$

whence $n \geqq N$ implies

$$f_n(x) - \tfrac{2}{3} = (-2)^{n-N}(f_N(x) - \tfrac{2}{3}).$$

Since $\lim_{n\to\infty} f_n(x)$ exists, then $f_N(x) = 2/3$. In the latter case we see that $n \geqq N$ implies

$$f_n(x) = 2^{n-N} f_N(x).$$

From the existence of $\lim_{n\to\infty} f_n(x)$, we see that $f_N(x) = 0$. We have then shown that if $\lim_{n\to\infty} f_n(x)$ exists, either $f_n(x) = 2/3$ for some n or $f_n(x) = 0$ for some n, or, equivalently (as is seen from the definition of f_n), either $x = j/2^n$ for some n and $j = 0, 1, \cdots, 2^n$ or $x = j/2^n \pm 1/3 \cdot 2^n$ for some n and $j = 0, 1, \cdots, 2^n$. Conversely: (a) if $x = j/2^n$ for some j and n, then $x = 2j/2^{n+1}$, whence $f_{n+1}(x) = f_n(x) = 0$ for all larger n, and thus $\lim_{n\to\infty} f_n(x) = 0$; (b) if $x = j/2^n \pm 1/3 \cdot 2^n$, then $f_{n+1}(x) = 2(1-f_n(x))$ and hence $f_{n+1}(x) = 2/3$ for all larger n, whence $\lim_{n\to\infty} f_n(x) = 2/3$.

We have now shown that $\lim_{n\to\infty} f_n(x)$ exists on the countable dense set of all points of the form $j/2^n \pm k/3 \cdot 2^n$, $n = 0, 1, \cdots$, $j = 0, 1, \cdots, 2^n$, $k = 0, 1$, and $\lim_{n\to\infty} f_n(x)$ does not exist elsewhere in the unit interval.

NOTE ON A PAPER OF SHANKS

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In his paper [4] Shanks proved the following formula

$$\binom{x}{i}^k = \sum_{s=1}^{ik-i+1} A_{ks}^{i} \binom{x+s-1}{ik}; \tag{1}$$

he showed that the A_{ks}^{i} are positive integers and that $A_{ks}^{i} = A_{k,ik-i-s+2}^{i}$, and also gave a determinantal formula for the A_{ks}^{i}.

In this note we shall make several remarks concerning (1). In particular we shall generalize formulas (3) and (4) below.

Consider first the case $i=1$; then (1) becomes

$$x^k = \sum_{s=1}^{k} A_{ks} \binom{x+s-1}{k}, \tag{2}$$

where $A_{ks} = A_{ks}^{1}$. This formula is due to Worpitzky [6] but the numbers A_{ks} had been introduced in a different manner by Euler [2; 487–491], who gave the explicit expression

$$A_{ks} = \sum_{r=0}^{s} (-1)^r \binom{k+1}{r} (s-r)^k \qquad (k \geqq 1). \tag{3}$$

The recursion formula

$$A_{k+1,s} = (k+2-s)A_{k,s-1} + sA_{ks} \tag{4}$$

is due to Worpitzky. An interesting formula of a different kind is quoted by Vandiver [5; 506]. Let

$$R_k = R_k(x) = \sum_{s=1}^{k} A_{ks} x^{s-1},$$

and put $H_k = R_k/(x-1)^k$, then

$$(H+1)^k = xH_k \qquad (k \geqq 1), \tag{5}$$

where the left member is to be expanded by the binomial theorem and superscripts degraded to subscripts.

Before returning to (1) we derive a formula which follows readily from some results due to Nielsen [3; 26–30]. Nielsen shows that

$$(-1)^{m+n} x^n = \sum_{s=0}^{m+1} \binom{x+s-1}{m} B_s^{m,n} \qquad (m \geqq n),$$

where

$$B_s^{m,n} = \sum_{r=0}^{s} (-1)^r \binom{m+1}{r} (s-r)^n.$$

Let $f(x) = \sum_{n=0}^{m} a_n x^n$ be an arbitrary polynomial of degree $\leqq m$. Then

$$\begin{aligned} f(x) &= \sum_{n=0}^{m} (-1)^{m-n} a_n \sum_{s=0}^{m+1} \binom{x+s-1}{m} B_s^{m,n} \\ &= \sum_{s=0}^{m+1} \binom{x+s-1}{m} \sum_{r=0}^{s} (-1)^r \binom{m+1}{r} \sum_{n=0}^{m} (-1)^{m-n} a_n (s-r)^n \\ &= (-1)^m \sum_{s=0}^{m+1} \binom{x+s-1}{m} \sum_{r=0}^{s} (-1)^r \binom{m+1}{r} f(r-s). \end{aligned}$$

Now

$$\sum_{r=0}^{m+1} (-1)^r \binom{m+1}{r} f(r-m-1) = 0,$$

since it is essentially the $(m+1)$th difference of a polynomial of degree $\leqq m$. We have therefore the following result. *Let $f(x)$ be an arbitrary polynomial of degree $\leqq m$; then we have*

$$f(x) = (-1)^m \sum_{s=0}^{m} C_s \binom{x+s-1}{m}, \tag{6}$$

where the coefficients C_s are determined by

$$C_s = \sum_{r=0}^{s} (-1)^r \binom{m+1}{r} f(r-s). \tag{7}$$

Note that when $f(0) = 0$, the term corresponding to $s = 0$ in (6) drops out.

If in (6) we take $f(x) = \binom{x}{i}^k$, $m \geqq ik$, then the right member of (7) reduces to

$$(-1)^{ki} \sum_{r=0}^{s} (-1)^r \binom{m+1}{r} \binom{s-r+i-1}{i}^k; \tag{8}$$

in particular, for $m = ik$ we get

$$A_{ks}^{i} = \sum_{r=0}^{s} (-1)^r \binom{ik+1}{r} \binom{s-r+i-1}{i}^k \qquad (i, k \geqq 1). \tag{9}$$

For $i = 1$, (9) clearly reduces to (3).

In order to extend (4) to the general case we require the following identity:

$$(x)_i = \sum_{j=0}^{i} \alpha_j (x+s+j-1)_j (x+s-\lambda-1)_{i-j}, \tag{10}$$

where $(x)_i = x(x-1)\cdots(x-i+1)$, $(x)_0 = 1$, and

$$\alpha_j = \binom{i}{j}\frac{(\lambda+1-s)_j(s+i-1)_{i-j}}{(\lambda+i)_i}. \tag{11}$$

This formula is readily proved by an induction with respect to i; λ is arbitrary, but for the application we take $\lambda = ik$. Then multiplying both sides of (1) by $\binom{x}{i}$ and using (10) we get

$$A^{i}_{k+1,s} = \frac{1}{i!}\sum_{j=0}^{i}\binom{i}{j}(ik-s+j+1)_j(s+i-j-1)_{i-j}A^{i}_{k,s-j}. \tag{12}$$

For $i=1$ this reduces to (4).

If we define the operator E by means of $Ef(s) = f(s-1)$ and agree that E operates only on s, we can put (12) in the form

$$A^{i}_{k+1,s} = \frac{1}{i!}\Omega_i A^{i}_{ks},$$

where

$$\Omega_i = \sum_{j=0}^{i}\binom{i}{j}(ik-s+j+1)_j(s+i-j-1)_{i-j}E^j. \tag{13}$$

Then Ω_i can be factored as follows:

$$\Omega_i = (\mu E+s)(\mu E+s+1)\cdots(\mu E+s+i-1), \tag{14}$$

where $\mu = ik-s+2$. It is easy to prove (14) by induction if we put it in slightly stronger form, namely, let

$$\Omega_i' = \sum_{j=0}^{i}\binom{i}{j}(\lambda-s+j+1)_j(s+i-j-1)_{i-j}E^j. \tag{15}$$

Then (14) holds for Ω_i' where now $\mu = \lambda-s+2$. Indeed it is only necessary to substitute from (15) in $\Omega_i'(E+s+i)$ and verify that the result reduces to Ω_{i+1}'.

We remark that the linear factors in (14) permute.

For formulas similar to (12) and (14) see [1].

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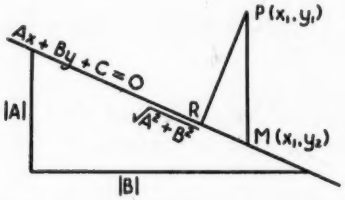
CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

DISTANCE FROM A LINE, OR PLANE, TO A POINT*

J. P. BALLANTINE and A. R. JERBERT, University of Washington



To find the perpendicular distance from the line $Ax+By+C=0$, to the point $P(x_1, y_1)$, we begin by computing the y-distance,

$$MP = y_1 - y_2 = y_1 - (-Ax_1 - C)/B = (Ax_1 + By_1 + C)/B, \tag{1}$$

where y_2 is the y-value obtained by substituting $x=x_1$ in the equation of the line. Since the y and x intercepts of the latter are in the ratio $(-C/B)/(-C/A) = A/B$, the intercept triangle has sides proportional to $|A|$, $|B|$, $\sqrt{A^2+B^2}$. MRP is evidently a similar triangle so that,

$$RP/MP = |B|/\sqrt{A^2 + B^2},$$

whence,

$$d = RP = \frac{|B|}{\sqrt{A^2 + B^2}} \cdot MP$$

$$= \frac{|B|(Ax_1 + By_1 + C)}{B\sqrt{A^2 + B^2}}, \qquad \text{by equation (1),}$$

$$= \frac{Ax_1 + By_1 + C}{(\text{signum } B)\sqrt{A^2 + B^2}} \qquad (\text{signum } B = |B|/B). \tag{2}$$

Any $K \neq 0$, which is multiplied into the coefficients A, B, C of the equation of the line evidently "divides out" of the factors in the right member of equation (2). These factors and their product d are therefore invariant under such a multiplication. The second factor MP, yields $+$ and $-$ values for points above and below the line, respectively, and this remains true for d since the first factor, $|B|\sqrt{A^2+B^2}$, is always positive.

* This paper is based on Professor J. P. Ballantine's treatment of the distance formula on page 235 of his book Essentials of Engineering Mathematics, published by Prentice Hall, 1938.

In three dimensions the x, y, and z distances from the plane $A_1x+A_2y+A_3z+A_4=0$, to the point $P(x_1, y_1, z_1)$ are given by the expressions,

$$(A_1x_1 + A_2y_1 + A_3z_1 + A_4)/A_i, \qquad i = 1, 2, 3,$$

respectively. To obtain the perpendicular distance we multiply by the cosine of the acute angle between the chosen axis and the normal to the plane, *i.e.* by $|A_i|/\sqrt{A_1^2+A_2^2+A_3^2}$. Thus, as before,

$$d = (A_1x_1 + A_2y_1 + A_3z_1 + A_4)/(\text{signum } A_i)\sqrt{A_1^2 + A_2^2 + A_3^2}.$$

In this manner d is $+$ or $-$ according as the point P is right or left, front or back, above or below, the plane, respectively.

A SIMPLE PROOF OF THE BINOMIAL THEOREM

C. M. FULTON, University of California, Davis

As most mathematics teachers know from experience, our freshman students find the usual formal proof of the binomial theorem hard to understand. In this paper we present a different proof whose simplicity, we feel, will speak for itself.

The tool for our proof will be the easily proven identity

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1}). \tag{1}$$

We use it first to write

$$(1 + x)^n - (1 + y)^n = (x - y)[(1 + x)^{n-1} + (1 + x)^{n-2}(1 + y) + \cdots + (1 + y)^{n-1}]. \tag{2}$$

Now, the students will grant the validity of the expansion

$$(1 + x)^n = 1 + {}_nC_1x + {}_nC_2x^2 + \cdots + {}_nC_nx^n \tag{3}$$

with coefficients ${}_nC_r$ to be determined, n being a positive integer. We can express the left side of (2) with the aid of (3) and apply (1) to the differences of like powers. Thus,

$$(1 + x)^n - (1 + y)^n = (x - y)[{}_nC_1 + {}_nC_2(x + y) + {}_nC_3(x^2 + xy + y^2) + \cdots + {}_nC_n(x^{n-1} + x^{n-2}y + \cdots + y^{n-1})]. \tag{4}$$

We now equate the right sides of (2) and (4) canceling the common factor $x-y$. In the resulting identity it is still permissible to make $x=y$. Hence,

$$n(1 + x)^{n-1} = {}_nC_1 + 2{}_nC_2x + 3{}_nC_3x^2 + \cdots + n{}_nC_nx^{n-1}. \tag{5}$$

On the other hand, because of (3), we also have

$$n(1 + x)^{n-1} = n(1 + {}_{n-1}C_1x + {}_{n-1}C_2x^2 + \cdots + {}_{n-1}C_{n-1}x^{n-1}). \tag{6}$$

On comparing the right sides of (5) and (6) we see that ${}_nC_1=n$. Then necessarily ${}_{n-1}C_1=n-1$. Since the coefficients of x are equal we find next ${}_nC_2=n(n-1)/2$.

Therefore, ${}_{n-1}C_2=(n-1)(n-2)/2$. This allows us to determine ${}_nC_3$, *etc.* In general,

$$ {}_nC_r = \frac{n}{r}\,{}_{n-1}C_{r-1}, \quad {}_nC_0 = 1 \qquad (r = 1, 2, \cdots, n). $$

Finally, from these recurrence formulas, we may easily prove, by induction, that

$$ {}_nC_r = \frac{n!}{r!(n-r)!} \qquad (r = 1, 2, \cdots, n). $$

INTEGRABILITY OF CONTINUOUS FUNCTIONS

M. J. Norris, College of St. Thomas

In many current calculus texts the proof that a continuous function is integrable is unsatisfactory; for it is assumed without mention, or at best stated to be true with proof deferred to advanced courses, that a function continuous on a closed interval is uniformly continuous there. Now we do not wish to advocate teaching uniform continuity to sophomores, but we would like to suggest another possible remedy in a different type of proof.

The method of this note can be used with any definition such that integrability of a function can be proved by showing that the function is bounded and that upper and lower integrals, satisfying (d) and (e) below, are equal. However, it seems particularly well suited to the definition of integrability suggested herein, which we feel is somewhat simpler than those involving generalized limit processes.

Given the closed interval $[a, b]$ we consider partitions, $P(x_0, x_1, x_2, \cdots, x_n)$, of $[a, b]$ into the n subintervals $[x_{i-1}, x_i]$, $(1 \leqq i \leqq n)$, where $a=x_0<x_1<x_2 \cdots <x_n=b$. We say that the partition P_2 is a refinement of the partition P_1 if every endpoint used in P_1 is also used in P_2. If now f is a bounded function on $[a, b]$, and P is a partition of $[a, b]$, we let $U(P)$ and $L(P)$ be respectively the sums $\sum_{i=1}^{n} M_i(x_i-x_{i-1})$ and $\sum_{i=1}^{n} m_i(x_i-x_{i-1})$, where M_i is the least upper bound of f on $[x_{i-1}, x_i]$ and m_i is the greatest lower bound of f there. It is easy to prove the following results:

(a) $L(P) \leqq U(P)$.

(b) If P_2 is a refinement of P_1, then $L(P_2) \geqq L(P_1)$ and $U(P_2) \leqq U(P_1)$.

(c) $L(P_1) \leqq U(P_2)$ for any two partitions P_1 and P_2.

We obtain (c) by taking a common refinement of P_1 and P_2 and then using (a) and (b).

It is now natural to consider the greatest lower bound, $\overline{\int_a^b} f dx$, of the $U(P)$'s and the least upper bound, $\underline{\int_a^b} f dx$, of the $L(P)$'s. Plainly we have:

(d) If M and m are the least upper bound and greatest lower bound, respectively, of f on $[a, b]$, then $m(b-a) \leqq \underline{\int_a^b} f dx \leqq \overline{\int_a^b} f\, dx \leqq M(b-a)$.

(e) If $a<c<b$, then $\underline{\int_a^c} f dx + \underline{\int_c^b} f dx = \underline{\int_a^b} f dx$ and $\overline{\int_a^c} f dx + \overline{\int_c^b} f dx = \overline{\int_a^b} f dx$.

We have (e) as a consequence of the facts that the values for $\underline{\int_a^b} f dx$ and $\overline{\int_a^b} f dx$

are completely determined by partitions which are refinements of $P(a, c, b)$ and that any partition of $[a, c]$ can be combined with any partition of $[c, b]$ to yield a partition of $[a, b]$.

Finally we say that f is integrable on $[a, b]$ if $\underline{\int}_a^b f dx = \overline{\int}_a^b f dx$. We are now ready to prove the theorem on integrability of continuous functions.

THEOREM. *If f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$.*

Proof. As a continuous function on a closed interval f is bounded on $[a, b]$. Thus we can define the function $Z(x)$ by

$$Z(x) = \overline{\int}_a^x f dx - \underline{\int}_a^x f dx \qquad (a < x \leqq b).$$

For $b - x \geqq h > 0$ we have

$$Z(x + h) - Z(x) = \overline{\int}_x^{x+h} f dx - \underline{\int}_x^{x+h} f dx.$$

Thus, if $M(h)$ and $m(h)$ are the maximum and minimum, respectively, of f on $[x, x+h]$, we have

$$0 \leqq Z(x + h) - Z(x) \leqq M(h)h - m(h)h,$$

$$0 \leqq \frac{Z(x + h) - Z(x)}{h} \leqq M(h) - m(h).$$

Now $\lim_{h \to 0} M(h)$ and $\lim_{h \to 0} m(h)$ both exist and are equal to $f(x)$, so $\lim_{h \to 0} [Z(x+h) - Z(x)]/h$ exists and is zero. A similar argument is available if $a - x < h < 0$, so $Z'(x)$ exists and is zero for $a < x < b$ and $Z(x)$ is continuous for $a < x \leqq b$. Thus $Z(x)$ is constant for $a < x \leqq b$; and as $\lim_{x \to a} Z(x)$ is clearly zero, we must have $Z(x)$ identically zero for $a < x \leqq b$. In particular, $\underline{\int}_a^b f dx = \overline{\int}_a^b f dx$.

Incidentally, the proof above is valid, after a few minor changes in wording, for functions which have right and left limits at every inner point of $[a, b]$, a right limit at a, and a left limit at b.

COORDINATE GEOMETRY FROM THE VECTOR POINT OF VIEW

SAMUEL BOURNE, University of Connecticut

Introduction. From a mathematical point of view, the vector approach to coordinate geometry is superior to the conventional slope approach, in that concepts do not depend on the dimensionality of the space, and proofs are simplified. From a practical point of view, its superiority rests in the early introduction of vectors into the student's mathematical education, thereby enabling him to utilize the algebra of vectors in his study of the physical sciences. The above opinions are supported by years of teaching this subject at the Johns

Hopkins University, under Professor F. D. Murnaghan. We give a few examples to illustrate them.

Vectors. In defining a vector, we are cognizant of the mathematical immaturity of the student and we constantly appeal to his geometric experience. If $P_1=(x_1, y_1)$ and $P_2=(x_2, y_2)$, we define the direction numbers of $\overrightarrow{P_1P_2}$ to be $\Delta x=x_2-x_1$ and $\Delta y=y_2-y_1$, and we write $\overrightarrow{P_1P_2}=(\Delta x, \Delta y)$. Conversely, each ordered pair of numbers determines directed line segments having these numbers as direction numbers. Hence, we may define a *vector as the collection of directed line segments having equal direction numbers.* This definition is adequate at this early date. Each directed line segment is called a representative line segment of the vector and the length of such a line segment is defined to be the magnitude of the vector. We denote by $\vec{\mathbf{v}}(P_1 \rightarrow P_2)$ a vector, a representative segment of which is $\overrightarrow{P_1P_2}$. If $\overrightarrow{P_1P_2}=(\Delta x, \Delta y)$ and $r=+\sqrt{(\Delta x)^2+(\Delta y)^2}\neq 0$, then we define the direction cosines (l, m) of P_1P_2 to be $l=\Delta x/r=\cos\theta$ and $m=\Delta y/r=\sin\theta$, where θ is an angle from the positive direction of the x-axis to the direction $\overrightarrow{P_1P_2}$. Obviously, $l^2+m^2=\cos^2\theta+\sin^2\theta=1$.

Angle. We let θ be an angle from the vector $\vec{\mathbf{v}}_1$ to the vector $\vec{\mathbf{v}}_2$; then we have

THEOREM 1. *If the direction cosines of* $\vec{\mathbf{v}}_1$ *are* (l_1, m_1) *and if the direction cosines of* $\vec{\mathbf{v}}_2$ *are* (l_2, m_2), *then* $\cos\theta=l_1l_2+m_1m_2$.

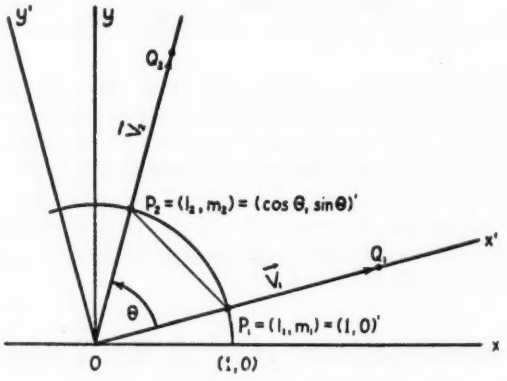


FIG. 1

We let $\overrightarrow{OQ_1}$ and $\overrightarrow{OQ_2}$ be representative segments of the vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, respectively, and P_1 and P_2 the points in which the unit circle, at the origin, intersects these segments. Then, $P_1=(l_1, m_1)$ and $P_2=(l_2, m_2)$ relative to xy-axes, and $P_1=(1, 0)$ and $P_2=(\cos\theta, \sin\theta)$, relative to $x'y'$-axes. We have

$$|\overrightarrow{P_1P_2}|^2=(l_2-l_1)^2+(m_2-m_1)^2=2-2(l_1l_2+m_1m_2),$$

$$|\overrightarrow{P_1P_2}|^2 = (\cos\theta - 1)^2 + \sin^2\theta = 2 - 2\cos\theta;$$

therefore

$$\cos\theta = l_1l_2 + m_1m_2.$$

Since

$$l_1 = \frac{\Delta_1 x}{r_1}, \quad l_2 = \frac{\Delta_2 x}{r_2}, \quad m_1 = \frac{\Delta_1 y}{r_1} \quad \text{and} \quad m_2 = \frac{\Delta_2 y}{r_2}, \quad r_1r_2 \neq 0,$$

we have that

$$\cos\theta = \frac{(\Delta_1 x)(\Delta_2 x) + (\Delta_1 y)(\Delta_2 y)}{r_1r_2}.$$

This equality motivates us to define *the symmetric product* $\overrightarrow{v_1} \cdot \overrightarrow{v_2}$ *of the vector* $\overrightarrow{v_1} = (x_1, y_1)$ *by the vector* $\overrightarrow{v_2} = (x_2, y_2)$ *to be* $x_1x_2 + y_1y_2$. We prove, with ease, the well-known properties of this product with their corresponding geometric interpretations.

Straight line. We use this symmetric product to prove

THEOREM 2. *An equation of a straight line is an equation of the form* $ax + by + c = 0$. *Conversely, the graph of the equation* $ax + by + c = 0$ *is a straight line.*

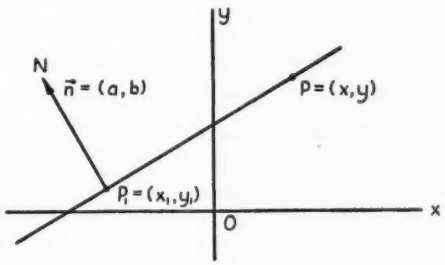


FIG. 2

We let $P_1 = (x_1, y_1)$ be a fixed point on the given line, $\overrightarrow{n} = (a, b)$ a vector normal to this line, and $P = (x, y)$ any point on this line. Then

$$\overrightarrow{v}(P_1 \rightarrow N) \cdot \overrightarrow{v}(P_1 \rightarrow P) = 0, \qquad P \neq P_1;$$

therefore

$$a(x - x_1) + b(y - y_1) = 0, \quad \text{for all} \quad P.$$

Therefore

$$ax + by + c = 0, \quad \text{where} \quad c = -ax_1 - by_1.$$

The proof of the converse follows by a retracing of steps.

Distance from a line to a point. Again, by a judicious employment of the symmetric product, we may prove

THEOREM 3. *The distance d from the line $ax+by+c=0$ to the point $P_1=(x_1, y_1)$ is given by*

$$\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}.$$

Proof:

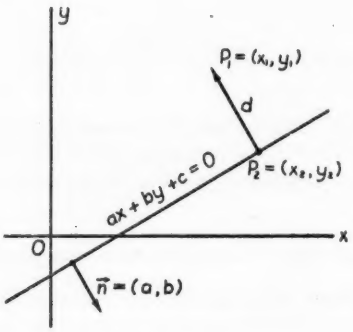


FIG. 3

We let $\vec{n}=(a, b)$ be a normal to the line $ax+by+c=0$ and $P_2=(x_2, y_2)$ the foot of the perpendicular from $P_1=(x_1, y_1)$ to this line. Then $\vec{n}$ and $\overrightarrow{P_1P_2}$ are parallel, and if θ is an angle from $\vec{n}$ to $\overrightarrow{P_1P_2}$ we have:

$$(a, b)\cdot(x_1-x_2, y_1-y_2)=\sqrt{a^2+b^2}\,d\cos\theta;$$

therefore

$$a(x_1-x_2)+b(y_1-y_2)=\pm d\sqrt{a^2+b^2},$$

and

$$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}.$$

Conclusion. The above few examples, in the plane, illustrate adequately the effectiveness of this approach. We may point out that no trigonometry need be presupposed. The definition of the vector product of two space vectors is motivated by the necessity to find direction numbers of the line of intersection of two planes. The laws of trigonometry are merely a by-product of the relationships of the algebra of plane vectors, while the laws of spherical trigonometry spring from the corresponding relationships of the algebra of space vectors.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1011. *Proposed by D. D. Wall, Los Angeles, California*

If traffic signals are "set for 30," at what other constant speeds may one travel? Assume signals evenly spaced, all change at the same time, and that they alternate in color spacewise as well as timewise.

E 1012. *Proposed by Oystein Ore, Yale University*

In a multiplication like

$$\begin{array}{r} 11 \\ \times 11 \\ \hline 11 \\ 11 \\ \hline 121 \end{array}$$

there occur only the two digits 1 and 2. Find all multiplications involving exactly two digits.

E 1013. *Proposed by P. A. Piza, San Juan, Puerto Rico*

Show that for $a=0, 1, 2, \cdots, 10$ and for any positive integral n except 1 and 3, the expression

$$(10^n - 1)(909a + 1)$$

yields palindromic numbers.

E 1014. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle having sides a, b, c and area S, if the ratio

$$(a^2 + b^2 + c^2)/4S$$

is a whole number $m>1$, then the sides a, b, c cannot all be integral.

E 1015. *Proposed by C. A. Shook and Albert Wilansky, Lehigh University*

Evaluate

$$\int \frac{dx}{x[J_n(x)]^2},$$

where J_n is the usual Bessel function.

SOLUTIONS

A Tetrahedron and Some Spheres

E 980 [1951, 564]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If a sphere through the vertices A, B, C of a tetrahedron $ABCD$ cuts the edges DA, DB, DC in A', B', C', then the distances of its center from those of the spheres $ABCD$ and $A'B'C'D$ are equal, respectively, to the radii of the spheres $A'B'C'D$ and $ABCD$

Solution by S. T. Thompson, Plattsburg, N. Y. The planes ABC and $A'B'C'$ are antiparallel with respect to the trihedral angle D. It follows that DO is perpendicular to $A'B'C'$ and DO' to ABC. But, if P is the center of the given sphere, PO' is perpendicular to $A'B'C'$ and PO to ABC. Thus $DOPO'$ is a parallelogram and the theorem is established. The analogous theorem for the plane is also true.

Also solved by Leon Bankoff and Joseph Langr.

Geometric Progression with Two Ratios

E 981 [1951, 564]. *Proposed by David Mandelbaum, Hillside, N. J.*

Derive a formula for the sum of the first n terms of a progression in which the first term is a, each even placed term is obtained from its preceding term by multiplying by the constant u, and each odd placed term (after the first) is obtained from its preceding term by multiplying by the constant v.

Solution by Ferrel Atkins, Bowling Green State University. The first n terms of the given series will be made up of $[(n+1)/2]$ terms of the series

$$a + a(uv) + a(uv)^2 + \cdots$$

and $[n/2]$ terms of the series

$$au + au(uv) + au(uv)^2 + \cdots,$$

where $[x]$ denotes the greatest integer not greater than x. Thus

$$S_n = a\left\{\frac{1-(uv)^{[(n+1)/2]}}{1-uv}\right\} + au\left\{\frac{1-(uv)^{[n/2]}}{1-uv}\right\}.$$

If $|uv|<1$, then

$$S_\infty = a(1+u)/(1-uv).$$

Also solved by A. N. Aheart, P. M. Anselone, Leon Bankoff, J. H. Braun, Fred Discepoli, F. F. Dorsey, D. H. Friedman, C. V. Fronabarger, B. K. Gold, Boyd Henry, Vern Hoggatt, J. M. Kingston, Sam Kravitz, Sidney Kravitz,

L. H. Lange, H. R. Leifer, Kovina Milosevich, R. Muguercia, Prasert Na Nagara, C. S. Ogilvy, Margaret Olmsted, C. F. Pinzka, Azriel Rosenfeld, C. W. Trigg, R. Z. Vause, Jr., G. W. Walker, E. J. Zirkel, and the proposer.

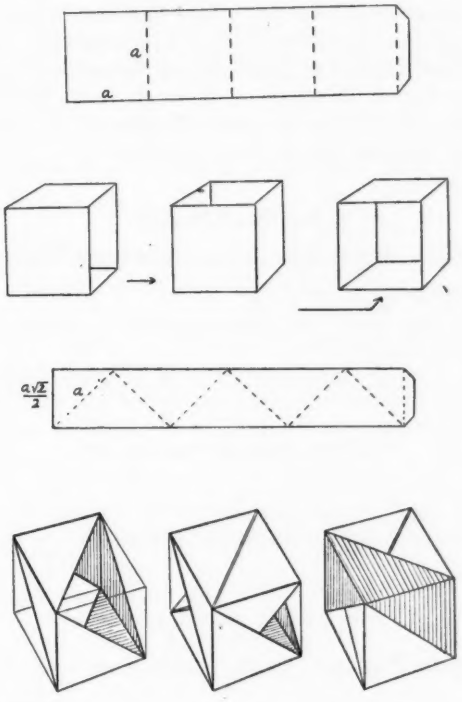
Milosevich pointed out that this sum was evaluated by L. Conte, *Rendite a termini variabili in progressione arithmetica o geometrica a due ragioni*, Giornale di Matematica Finanziaria, vol. VII, No. 1–2, 1949, pp. 27–46.

Folding Cylinders into a Cube

E 982 [1951, 564]. *Proposed by C. W. Trigg, Los Angeles City College*

Show that there are two different sets of three congruent right cylindrical surfaces which may be folded and assembled into a cube with no open edges.

Solution by Leon Bankoff, Los Angeles, California. The following diagrams are self-explanatory.



Also solved by the proposer.

A Geometric Inequality

E 983 [1951, 564]. *Proposed by A. W. Goodman, University of Kentucky*

Let M and N be two points one unit apart. With M and N as centers and with unit radii draw arcs ANB and AMB. Let Q be any point on arc AMB and P_1 and P_2 any points on arc ANB such that N is the midpoint of arc P_1P_2. Show that

$$QP_1 + QP_2 \leqq 2 \leqq (QP_1)^2 + (QP_2)^2.$$

Solution by C. W. Trigg, Los Angeles City College. Let O be the midpoint of chord P_1P_2. The locus of Q' such that $Q'P_1+Q'P_2=2$ is an ellipse with semi-minor axis OM (since $P_1M+P_2M=2$), and major axis lying along P_1P_2, centered at O and equal to 2. Now O lies between M and N, so the arc AMB lies within the ellipse and is tangent to it at M. Hence $QP_1+QP_2 \leqq Q'P_1+Q'P_2=2$.

The locus of Q'' such that $(Q''P_1)^2+(Q''P_2)^2=2$ is a circle with its center at O, and radius OM (since $(P_1M)^2+(P_2M)^2=2$). Now $OM<NM$, so the circle $O(M)$ lies wholly within the circle $N(M)$ and is tangent to $N(M)$ at M. Hence $2=(Q''P_1)^2+(Q''P_2)^2 \leqq (QP_1)^2+(QP_2)^2$. Hence the proposition.

Also solved by Ferrel Atkins, Leon Bankoff, J. H. Braun, Fred Discepoli, C. V. Fronabarger, W. J. Klimczak, Sam Kravitz, F. M. Morgan, Prasert Na Nagara, Margaret Olmsted, and R. P. Tapscott.

An Iterated Function

E 984 [1951, 564]. *Proposed by Joseph Rosenbaum, Hartford, Conn.*

(a) Find $f(x)$ when $f[f(x)]=x^2-2$.

(b) More generally, find $f_1(x)$ when $f_n(x)=x^2-2$, where $f_n(x)$ is defined by the relation $f_{r+1}(x)=f_1[f_r(x)]$.

I. *Solution by J. L. Botsford and R. E. Wild, University of Idaho.* For the general case take

$$f_1(x) = 2 \cos (2^{1/n} \arccos x).$$

Then

$$f_r(x) = 2 \cos (2^{r/n} \arccos x),$$

and

$$f_n(x) = 2 \cos (2 \arccos x) = x^2 - 1.$$

II. *Solution by the Proposer.* For the general case take

$$f_1(x) = [(x + \sqrt{x^2 - 4})/2]^{2^{1/n}} + [(x - \sqrt{x^2 - 4})/2]^{2^{1/n}}.$$

The desired relation may be conveniently verified by making the substitution $x=r+1/r$.

A Problem in Partitioning

E 985 [1951, 564]. *Proposed by Leo Moser, Texas Technological College*

What is the smallest integer N such that if the integers $1, 2, 3, \cdots, N$ are distributed into three classes in any manner whatsoever at least one of the classes will contain a solution of $a+b=c$?

I. *Solution by G. W. Walker, Buffalo, N. Y.* The partitioning

1, 2, 4, 8, 11, 22

3, 5, 6, 7, 19, 21, 23

9, 10, 12, 13, 14, 15, 16, 17, 18, 20

shows that $N>23$. We might also have either 16 or 17 (but not both) in the first row instead of in the third. There are just these three partitionings of the first 23 positive integers into three rows without a case of $a+b=c$ in some row. In none of these cases can we add 24 to any row. Hence $N=24$.

If N_s is the smallest integer that cannot by any arrangement be accommodated in s rows, then $N_4=67$ and $N_5=197$. It can be shown that

$$2N_s < N_{s+1} < 3N_s.$$

Also solved similarly by J. H. Braun, who claimed that

$$N_s \geqq 3(3^s + 2s - 1)/4.$$

II. *Remark by the Proposer.* In the above solution it is assumed, in the relation $a+b=c$, that a, b, c are distinct. Without this limitation we may show that $N \leqq 16$. For, if the numbers $1, 2, \cdots, 16$ are put into three classes A, B, C, then at least one of these, say A, will contain at least six numbers, say $a_1<a_2<a_3<a_4<a_5<a_6$. Now consider the five numbers $b_1=a_6-a_5$, $b_2=a_6-a_4$, $b_3=a_6-a_3$, $b_4=a_6-a_2$, $b_5=a_6-a_1$. We may assume that none of these is in A, for otherwise A would contain a solution to $a+b=c$. We may therefore suppose that at least three of these are in B or C. Suppose $b_1'<b_2'<b_3'$ are three b's in B. Next consider $c_1=b_3'-b_2'$, $c_2=b_3'-b_1'$. If c_1 or c_2 is in A or B, then this class would contain a solution to $a+b=c$. Hence we may assume that c_1 and c_2 are in C. Finally, consider c_2-c_1. If this number is in A, B, or C, then that class would contain a solution to $a+b=c$.

On the other hand, the partitioning

1, 4, 10, 13

2, 3, 11, 12

5, 6, 7, 8, 9

shows that $N \geqq 14$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4483. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Find the sum of

$$1+\frac{1}{2}\,\frac{1}{3^2}+\frac{1\cdot 3}{2\cdot 4}\,\frac{1}{5^2}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{1}{7^2}+\cdots.$$

4484. *Proposed by Murray Gerstenhaber and R. S. Varga, Harvard University*

Given a real number $y>0$. Let $g_n=h_n/n$ be the rational number which best approximates y. where h_n is a positive integer and $n=1, 2, 3, \cdots$. Let

$$\lambda_n=\begin{cases}+ & \text{if } \ h_n/n-y\geqq 0,\\ - & \text{if } \ h_n/n-y<0.\end{cases}$$

Show that λ_n is periodic, *i.e.* there exists a positive integer k such that λ_{mk} is of one sign for all positive integers m, if and only if y is a positive rational number.

4485. *Proposed by Simon T. Kao, Catholic University of America, Washington, D. C.*

Through the orthocenter H of a triangle ABC draw any pair of perpendicular lines l_1 and l_2, and let A_1, A_2; B_1, B_2; C_1, C_2 be the respective points of intersection with the three sides BC, CA, AB. Show that the three points P, Q, R which divide the three segments A_1A_2, B_1B_2, C_1C_2 in the same ratio r, lie on a line. This is a generalization of the Droz-Farny theorem.

4486. *Proposed by D. J. Newman, Harvard University*

Let $f(x)$ be any function continuous on the positive axis. Prove that there exists an entire function which is always larger than this $f(x)$ along the positive axis. (That is, prove that there are arbitrarily large entire functions.)

4487. *Proposed by Paul Erdös, University College, London, England*

Let

$$\sum_{k=1}^{\infty} d_k=\infty, \qquad d_k>0, \qquad D_n=\sum_{k=1}^{n} d_k;$$

$$\sum_{k=1}^{\infty} a_k < \infty, \qquad a_k > 0.$$

Further assume that $a_k/d_k \geqq a_{k+1}/d_{k+1}$. Prove

$$\lim_{k\to\infty} \frac{a_k D_k}{d_k} = 0.$$

SOLUTIONS

Sets Having a Divisor Property

4358 [1949, 480]. *Proposed by Paul Erdös, University College, London, England*

Let $a_1 < a_2 < \cdots$ be a sequence of integers with the property that there does not exist an infinite subsequence of the a's in which no one divides another. Prove that the products

$$a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k}, \qquad 0 \leqq \alpha_i$$

have the same property.

If the a's are chosen to be the first k primes, then we obtain a well-known result due to Dickson.

Solution by Paul Erdös and Richard Rado, University College, and King's College, London. Let S be a set of positive integers. We say that S has the *divisor property* if every infinite subset of S contains two distinct numbers one of which divides the other. The present problem may be restated as the

THEOREM. *If S possesses the divisor property, and if S' is the set of all numbers of the form $a_1 a_2 \cdots a_n$, where $n > 0$, $a_\nu \in S$, then S' possesses the divisor property.*

The letters μ, ν, ρ, k, m denote positive integers, S is a fixed set of positive integers greater than 1, and the letter a denotes elements of S. By S^* we denote the set of all numbers of the form $a_1 a_2 \cdots a_n$, where $n \geqq 0$. Thus $1 \in S^*$. The letters A, B, C, D, E, F denote elements of S^*. We assume that S has the divisor property and proceed to show that S^* has the divisor property.

LEMMA.† *If $a_\nu \in S$ for all ν, then there are indices ν_ρ such that $\nu_1 < \nu_2 < \cdots$, $a_{\nu_1} | a_{\nu_2} | \cdots$.*‡

Proof. Let M be the set of all μ such that $a_\mu > a_\nu$ if $\mu < \nu$. Then M is finite, since otherwise we could find $\mu_\rho \in M$ such that $\mu_1 < \mu_2 < \cdots$. Then $a_{\mu_1} > a_{\mu_2} > \cdots$, which is impossible. Now we can choose an index μ_1 greater than all elements of M and, successively, μ_2, μ_3, $\cdots$ such that $\mu_1 < \mu_2 < \cdots$, $a_{\mu_1} \leqq a_{\mu_2} \leqq \cdots$. Put $M_1 = \{\mu_1, \mu_2, \cdots\}$. Let M_2 be the set of all $\mu \in M_1$ such that a_μ is

† This is problem 4330 [1950, 493–494].

‡ The symbol $x|y|z| \cdots$ means $x|y$, $y|z$, $\cdots$.

not a divisor of a_ν if $\mu<\nu$, $\nu\in M_1$. Then M_2 is finite. For, otherwise, since S has the divisor property, we could find distinct indices μ, $\nu\in M_2$ such that $a_\mu \mid a_\nu$. Then, by definition of M_2, $\nu<\mu$ and therefore, by definition of M_1, $a_\nu \leqq a_\mu$, $a_\nu = a_\mu$, $a_\nu \mid a_\mu$, which contradicts $\nu\in M_2$. Now we can choose $\nu_1\in M_1$, ν_1 greater than all elements of M_2, and, successively, $\nu_2, \nu_3, \cdots \in M_1$ such that $\nu_1<\nu_2 < \cdots$, $a_{\nu_1} \mid a_{\nu_2} \mid \cdots$, and the lemma is proved.

Proof of the theorem. Let $A_\nu\in S^*$ for all ν. Our aim is to find distinct indices μ, ν such that $A_\mu \mid A_\nu$. We define, inductively, numbers n_k, B_k, C_k as follows:

(i) We put $n_0=1$, $B_0=1$, $C_0=A_1$. We note that it is possible to choose m_ρ, D_ρ, E_ρ such that $n_0<m_1<m_2< \cdots$,

$$A_{m_\rho} = D_\rho E_\rho, \qquad B_0 \mid D_1 \mid D_2 \mid D_3 \mid \cdots,$$

a possible choice being $m_\rho=\rho+1$, $D_\rho=1$, $E_\rho=A_{\rho+1}$.

(ii) Suppose that we have already defined, for some k, numbers n_{k-1}, B_{k-1}, C_{k-1} in such a way that it is possible to choose numbers m_ρ', D_ρ', E_ρ' such that $n_{k-1}<m_1'<m_2'< \cdots$, $A_{m'_\rho}=D_\rho' E_\rho'$, $B_{k-1} \mid D_1' \mid D_2' \mid \cdots$. Then we choose m_ρ', D_ρ', E_ρ' in such a way that E_1' is least possible, and we put $n_k=m_1'$, $B_k = D_1'$, $C_k=E_1'$.

The numbers n_k, B_k, C_k have now been defined for all k, and they satisfy the following conditions:

$$n_1 < n_2 < n_3 < \cdots, \qquad A_{n_\rho} = B_\rho C_\rho, \qquad B_1 \mid B_2 \mid \cdots.$$

Moreover, for every k, the number C_k has the following *minimum property*. Whenever $n_{k-1}<m_1''<m_2''< \cdots$, $A_{m''_\rho}=D_\rho'' E_\rho''$, and $B_{k-1} \mid D_1'' \mid D_2'' \mid \cdots$, then $C_k \leqq E_1''$.

(iii) We now distinguish two cases.

Case 1. Suppose that there is a k such that $C_k=1$. Then $A_{n_k}=B_k \mid B_{k+1} \mid A_{n_{k+1}}$, and the desired result holds for $\mu=n_k$, $\nu=n_{k+1}$.

Case 2. Suppose that $C_k>1$ for every k. Then there are numbers a_k, F_k such that $C_k=a_k F_k$. By the lemma, there are indices ν_ρ such that $\nu_1<\nu_2< \cdots$, $a_{\nu_1} \mid a_{\nu_2} \mid \cdots$. Then

$$A_{n_{\nu_\rho}} = B_{\nu_\rho} C_{\nu_\rho} = (B_{\nu_\rho} a_{\nu_\rho}) F_{\nu_\rho}, \qquad n_{\nu_1-1} < n_{\nu_1} < n_{\nu_2} < \cdots,$$
$$B_{\nu_1-1} \mid B_{\nu_1} a_{\nu_1} \mid B_{\nu_2} a_{\nu_2} \mid \cdots, \qquad F_{\nu_1} = C_{\nu_1}/a_{\nu_1} < C_{\nu_1},$$

which contradicts the minimum property of C_{ν_1}. This completes the proof of the theorem.

Note. The following more general theorem can be proved. Let $\{a_\alpha\}$ be a partially ordered set with the property that every infinite subset $\{a_{\alpha_k}\}$ contains a finite subset $b_1, b_2, \cdots, b_l$ such that every a_{α_k} is no less than at least one of the b's. A set with this property may be called partially well ordered. (For a well ordered set $l=1$.)

Consider now the set of all vectors of finite length

$$(a_{\alpha_1}, a_{\alpha_2}, \cdots, a_{\alpha_n})$$

formed from the a's. We define

$$(a_{\alpha_1}, a_{\alpha_2}, \cdots, a_{\alpha_r}) \leqq (a_{\beta_1}, \cdots, a_{\beta_l})$$

if there exists a one to one mapping of the a_α's into the a_β's so that each a_α is less than or equal to the corresponding a_β. Our theorem now states that these vectors form a partially well ordered set. The theorem of problem 4358 is the special case in which the partial order in the set of integers is defined by $a \leqq b$ if $a \mid b$.

G. Higman and B. H. Neumann have found, independently of each other and of the present writers, the following still more general theorem: The set of vectors of finite length remains partially well ordered if we define

$$(a_{\alpha_1}, \cdots, a_{\alpha_r}) \leqq (a_{\beta_1}, \cdots, a_{\beta_l})$$

if there exists a mapping $f(x)$ of the integers $1, 2, \cdots, r$ into the integers $1, 2, \cdots, l$ so that $f(x)$ is monotone increasing and

$$a_{\alpha_j} \leqq a_{\beta_{f(j)}}.$$

Subdivisions of an Interval

4411 [1950, 564]. *Proposed by L. C. Hsu, National Tsing-Hua University, Peiping, China*

Show that if α is a real number >0, then for any given $\epsilon>0$ and any integer $k>0$ there can be found a number $\delta=\delta(\epsilon)>0$ and an integer $N \geqq k$ such that, if

$$\Delta x_1 + \Delta x_2 + \cdots + \Delta x_N = \Delta x_1' + \Delta x_2' + \cdots + \Delta x_N' = \alpha,$$
$$S_k = \sum \Delta x_{\nu_1} \cdots \Delta x_{\nu_k}, \qquad S_k' = \sum \Delta x_{\nu_1}' \cdots \Delta x_{\nu_k}',$$

then

$$|S_k - S_k'| < \epsilon$$

whenever $0<\Delta x_i<\delta$, $0<\Delta x_i'<\delta$, where S_k and S_k' are extended over all possible terms with distinct subscripts $\nu_1, \cdots, \nu_k$.

Solution by Morgan Ward, Pasadena, California. With the Proposer's notation we shall prove that if K is any assigned positive integer, for a suitable $N \geqq K$ and a suitable $\delta>0$,

$$|S_k - S_k'| < \epsilon \qquad k = 1, 2, \cdots, K, \tag{1}$$

provided that the Δx, $\Delta x'$ are in absolute value less than δ.

Choose N for the present merely larger than K, and let

$$\sigma_r = \sum \Delta x_i^r; \qquad \sigma_r' = \sum \Delta x_i'^r, \qquad (r = 1, 2, \cdots, k)$$

be the Newton power sums of the Δx; $\Delta x'$. Then S_k; S_k' are the elementary symmetric functions of the Δx; $\Delta x'$. Hence

$$S_1 = \sigma_1 = S_1' = \sigma_1' = \alpha, \tag{2}$$

and in general by Newton's formulas for $2 \leqq k \leqq K$,

$$\begin{aligned} S_k &= \frac{1}{k!}\left\{\sigma_1^k + \sum C_{(r)}\sigma_1^{r_1}\sigma_2^{r_2}\cdots\sigma_k^{r_k}\right\} \\ S_k' &= \frac{1}{k!}\left\{\sigma_1'^k + \sum C_{(r)}\sigma_1'^{r_1}\sigma_2'^{r_2}\cdots\sigma_k'^{r_k}\right\}. \end{aligned} \tag{3}$$

Here the $C_{(r)}$ are integers, and the sum is extended over all $r_i \geqq 0$ such that $r_1+2r_2+3r_3+\cdots+kr_k=k$. Furthermore, $r_1<k$ for every term in the summation, so that

$$r_2 + r_3 + \cdots + r_k \geqq 1. \tag{4}$$

Assign any positive ϵ less than 1, and let

$$M = \max \frac{\alpha^k}{k!}\sum |C_{(r)}|, \qquad k = 2, 3, \cdots, K.$$

Now choose the positive integer N so that both $N \geqq K$ and

$$2\alpha M/N < \epsilon. \tag{5}$$

Let $\delta = \alpha/N$. Then

$$N\delta = \alpha, \qquad 0 < \delta < 1. \tag{6}$$

Now if all the $|\Delta x|$, $|\Delta x'|$ are less than δ,

$$|\sigma_r|, \ |\sigma_r'| < N\delta^r = \alpha\delta^{r-1}, \qquad r = 2, 3, \cdots, K. \tag{7}$$

Hence formulas (3), (2) and (1) give the inequality

$$|S_k - S_k'| < \frac{2}{k!}\sum |C_{(r)}|\,\alpha^k\delta^{r_2+2r_3+\cdots+(k-1)r_k} < 2M\delta = \frac{2\alpha M}{N} < \epsilon$$

by (4), (6) and (5). Since (1) is trivial if $k=1$, the proof is complete.

The proof is readily modified to show that if we subject the S_k to L fixed constraints:

$$S_k = S_k' = \alpha_k, \qquad k = 1, 2, \cdots, L,$$

then the inequalities (1) hold for any fixed K, provided merely that $\alpha_1=\alpha$ is positive.

Also solved by O. E. Stanaitis and the Proposer.

Probability, Disjunct Events

4412 [1950, 635]. *Proposed by Jacques Dutka, Rutgers University*

Of the freshmen admitted to a certain college last fall it was found that:

74% were seventeen years of age or older,
89% weighed 125 pounds or more,
78% measured $5\frac{1}{2}$ feet tall or more.

Let p be the probability that a freshman chosen at random had all three characteristics. Determine bounds for the value of p, and generalize the problem.

Solution by C. F. Pinzka, Xavier University, Cincinnati. Consider the generalization in which a fraction f_i of the students have the property P_i $(i=1, 2, \cdots, n)$. Since the fraction not having the property P_i is $(1-f_i)$, the minimum fraction having all the properties is $1-\sum(1-f_i)$ or 0, whichever is greater, and which we denote by

$$\max\left[1-\sum(1-f_i), 0\right] = \max\left[\sum f_i - (n-1), 0\right].$$

The maximum fraction having all the properties is the minimum of the f_i, which we denote by $\min(f_i)$. Thus

$$\max\left[\sum f_i - (n-1), 0\right] \leqq p \leqq \min(f_i).$$

For the numerical example given, $0.41 \leqq p \leqq 0.74$.

This analysis implies that the properties are logically disjunct. That is, it is possible for a student to have any given subset of the properties and lack the remaining properties.

Also solved by Ferrel Atkins, N. J. Fine, G. P. Henderson, Charles McCracken, Jr., O. Dale Smith, S. L. Thompson, and the Proposer.

Editorial Note. Smith finds the problem in Birkhoff and MacLane, *Modern Algebra*, pp. 324–325. Azriel Rosenfeld refers to the solution to Knot X (#1) in Lewis Carroll, *A Tangled Tale* (Complete works of Lewis Carroll, Modern Library, p. 1071.)

Number of Integers of Special Form

4413 [1950, 635]. *Proposed by Paul Erdös, University College, London, England*

Let $a_1=2\cdot 3$, $a_2=3\cdot 5$, $a_3=5\cdot 7$, $\cdots$, $a_k=p_k\cdot p_{k+1}$, $\cdots$, where p_k is the kth prime. Denote by $f(x)$ the number of integers $\leqq n$ composed entirely of the a's (*i.e.*, the integers of the form $\prod a_i^{\alpha_i}$, $0\leqq\alpha_i$). Prove that

$$f(n) = cn^{1/2} + o(n^{1/2}), \tag{1}$$

where $\frac{1}{2}<c<1$.

Solution by the Proposer. It will turn out that

$$c = \prod_{i=1}^{\infty} \frac{1-p_i^{-1}}{1-(p_ip_{i+1})^{-1/2}}. \tag{2}$$

(It is easy to see that the product for c converges. In fact it remains convergent

if $p_1<p_2<\cdots$ is replaced by any increasing sequence of real numbers.)

Let k be a large integer. Denote by $1=A_1(k)<A_2(k)<\cdots$ the integers composed of $a_1, a_2, \cdots, a_k$; and by $B_1(k)<B_2(k)<\cdots$ the integers composed of $a_{k+1}, a_{k+2}, \cdots$. We shall put A_i, B_i for $A_i(k)$, $B_i(k)$. Denote by $g_k(m)$ the number of B's which do not exceed m. Clearly

$$f(n) = \sum_{i=1}^{\infty} g_k(n/A_i). \tag{3}$$

Denote by $h_j(n)$ the number of integers $\leqq n$ composed of $p_{j+1}^2, p_{j+2}^2, \cdots$; *i.e.*, $h_j(n)$ equals the number of squares $k^2 \leqq n$ for which p_i does not divide k, $1 \leqq i \leqq j$. We have

$$h_{j+1}(m) \leqq g_j(m) \leqq h_j(m); \tag{4}$$

the first inequality of (4) follows by letting p_k^2 correspond to a_k, $k=j+1, j+2, \cdots$; the second inequality follows by letting a_k correspond to p_{k+1}^2. From (4), with $j=0$, we have $\frac{1}{2}n^{1/2}<f(n)<n^{1/2}$.

From (3) and (4) we have

$$\sum_{i=1}^{\infty} h_{k+1}(n/A_i) \leqq f(n) \leqq \sum_{i=1}^{\infty} h_k(n/A_i). \tag{5}$$

By the sieve of Eratosthenes we have

$$n^{1/2}\prod_{i=1}^{k}(1-1/p_i) - 2^k < h_k(n) < n^{1/2}\prod_{i=1}^{k}(1-1/p_i) + 2^k. \tag{6}$$

Also trivially,

$$h_k(m) \leqq m^{1/2}. \tag{7}$$

Let t be large but fixed (*i.e.*, independent of n). From (5) and (7),

$$\sum_{i=1}^{t} h_{k+1}(n/A_i) < f(n) < \sum_{i=1}^{t} h_k(n/A_i) + n^{1/2}\sum_{i=t+1}^{\infty} 1/A_i^{1/2}.$$

Using (6) we obtain

$$\begin{aligned} n^{1/2}\prod_{i=1}^{k+1}(1-1/p_i)\sum_{i=1}^{t} 1/A_i^{1/2} - 2^{k+1}t < f(n) \\ < n^{1/2}\prod_{i=1}^{k}(1-1/p_i)\sum_{i=1}^{t} 1/A_i^{1/2} + 2^k t + n^{1/2}\sum_{i=t+1}^{\infty} 1/A_i^{1/2}. \end{aligned} \tag{8}$$

Now $\sum 1/A_i^{1/2}$ clearly converges. Furthermore

$$\lim_{k\to\infty}\prod_{i=1}^{k}(1-1/p_i)\sum_{i=1}^{\infty} 1/A_i^{1/2}$$

exists. In fact

$$\sum_{i=1}^{\infty} \frac{1}{A_i^{1/2}} = \prod_{i=1}^{k} \left[1 + \frac{1}{(p_i p_{i+1})^{1/2}} + \frac{1}{p_i p_{i+1}} + \frac{1}{(p_i p_{i+1})^{3/2}} + \cdots\right]$$

$$= \prod_{i=1}^{k} \frac{1}{1 - (p_i p_{i+1})^{-1/2}}.$$

Thus with c given by (2), choosing t and k large enough, we have

$$|f(n) - cn^{1/2}| < \epsilon n^{1/2}, \qquad n > n_0,$$

which is the desired relation (1).

Circumscribed Steiner Ellipse of a Triangle

4414 [1950, 635]. *Proposed by Lucien Droussent, Clermont Ferrand, France*

The trilinear polar, with respect to a triangle T, of any point P on the circumscribed Steiner ellipse of T is tangent to the conjugate conic of T that has P for center. (The circumscribed Steiner ellipse of a triangle ABC is the circumscribed ellipse whose tangents at A, B, C are parallel to the opposite sides BC, CA, AB.)

Solution by the Proposer. It is convenient to use barycentric coördinates so that the centroid G of the fundamental triangle T is (1, 1, 1), the line at infinity is $\alpha+\beta+\gamma=0$, and the circumscribed Steiner ellipse has the equation $\beta\gamma+\gamma\alpha+\alpha\beta=0$ and has G for its center. If $\alpha_0, \beta_0, \gamma_0$ are the coördinates of the point P, then

$$\frac{1}{\alpha_0} + \frac{1}{\beta_0} + \frac{1}{\gamma_0} = 0. \tag{1}$$

The trilinear polar of P has for its equation

$$\frac{\alpha}{\alpha_0} + \frac{\beta}{\beta_0} + \frac{\gamma}{\gamma_0} = 0 \tag{2}$$

and hence evidently passes through G. The tangential equation of the conjugate conic of T with center P is

$$\alpha_0 u^2 + \beta_0 v^2 + \gamma_0 w^2 = 0. \tag{3}$$

The proof is completed by observing that, by virtue of (1), the coördinates $(1/\alpha_0, 1/\beta_0, 1/\gamma_0)$ of the line (2) satisfy (3).

Integers Having Common Factors with Members of a Sequence

4416 [1950, 691]. *Proposed by D. J. Newman, Harvard University*

Let $a_1, a_2, \cdots$ be a sequence of integers such that $\sum_{n=1}^{\infty} 1/a_n$ diverges.

Show that almost all integers have a factor in common with some a_n.

Solution by Samuel Skolnik, Los Angeles City College. Let $\{r_i\}$ be the set of all prime factors of a_n $(n=1, 2, \cdots)$. For any N we have

$$\prod_{r_i \leqq a_N} \left(\frac{1}{1-1/r_i}\right) = \prod_{r_i \leqq a_N} \left(1+\frac{1}{r_i}+\frac{1}{r_i^2}+\cdots\right) \geqq \sum_{a_N} \frac{1}{a_n},$$

where the last sum is taken over all a_n none of whose prime factors exceeds a_N. This condition is satisfied by all $a_n \leqq a_N$ so that

$$\prod_{r_i \leqq a_N} \left(\frac{1}{1-1/r_i}\right) \geqq \sum_{n=1}^{N} \frac{1}{a_n},$$

and since $a_N \to \infty$ implies $N \to \infty$, and since $\sum_{n+1}^{\infty} 1/a_n$ diverges, we conclude that $\prod_i (1-1/r_i)^{-1}$ diverges; hence $\prod_i (1-r_i)$ diverges to zero.

Let $r_1, r_2, \cdots, r_2$ be the first k primes, in the order of magnitude, of the set $\{r_i\}$. The number of integers not greater than n, and not divisible by any of the primes $r_1, \cdots, r_k$ is

$$f(n, k) = [n] - \sum_i [n/r_i] + \sum_{i,j} [n/r_i r_j] - \cdots, \tag{1}$$

where $i, j, \cdots$ are unequal and run from 1 to k, and where $[x]$ indicates the integral part of x. The total number of brackets in (1) is $1+k+k(k-1)/2 + \cdots = 2^k$, and since $0 \leqq x-[x]<1$ we have

$$f(n, k) \leqq n - \sum_i n/r_i + \sum_{i,j} n/r_i r_j - \cdots + 2^k = n \prod_{r_i \leqq r_k} (1-1/r_i) + 2^k.$$

Let $f(n)$ denote the number of integers less than n, which are not divisible by any of the primes r_i. Then

$$f(n) \leqq f(n, k) \leqq n \prod_{r_i \leqq r_k} (1-1/r_i) + 2^k.$$

But $\prod_i (1-1/r_i)$ diverges to zero and hence, for arbitrary ϵ and for k sufficiently large we have $\prod_{r_i \leqq r_k} (1-1/r_i) < \frac{1}{2}\epsilon$, and for n sufficiently large, $2^k/n < \frac{1}{2}\epsilon$, whence $f(n) < \epsilon n$ and

$$\lim_{n \to \infty} \frac{f(n)}{n} = 0$$

which establishes the theorem.

It was assumed throughout the foregoing that the members of the sequence $\{a_n\}$ are distinct. The theorem is evidently false if, *e.g.*, $a_n = 2$, $(n=1, 2, \cdots)$. The notation and techniques employed are those of Hardy and Wright, *An Introduction to the Theory of Numbers*, pp. 233, 338, 341.

Solved also by the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Secondary Mathematics, A Functional Approach for Teachers. By H. F. Fehr, Boston, D. C. Heath and Company, 1951. 11+431 pages. $4.25.

This book has been written for teachers and prospective teachers of secondary school mathematics. It is a mathematics book and not a methods text. Rather the author presents, in a form suitable for use as a textbook, topics in mathematics which have a clearly determined relationship to the content of secondary school mathematics and the study of which will broaden the teacher's knowledge of the mathematics he will teach. The book can be recommended also for the professional library of secondary school teachers in service as a book which will renew and increase their interest in more advanced mathematics and insight into topics in many cases previously studied.

It is intended that, as a text, this book be used to supplement existing courses in teacher education programs. The materials in the book have been used by the author over a period of twelve years in classes for seniors and graduate students at Montclair State Teachers College and Teachers College, Columbia University. In the preface the author states that his book has, in part, the same purpose for secondary school mathematics teachers in the United States that Felix Klein's three volumes on *Elementary Mathematics From An Advanced View-Point* had for teachers in Germany. The author also expresses his desire to continue the pioneer work in America of his former teacher and friend, John A. Swenson, who was an early leader in this country in the teaching of advanced mathematics, with special reference to the implications and relationships of the content to the topics of secondary school mathematics.

One indication of the wide choice of topics presented is given by such chapter headings as Congruence in Geometry, Elementary Curve Tracing, The Binomial Series and the Exponential Function, the Derivative, Denumerability and Non-denumerability, Complex Exponents and Roots, Three-Dimensional Geometry, Hyperbolic Functions, and Geometric Constructions with Compasses and Straightedge. One of the sources of strength of this book is the selection of topics, all important for the teacher and yet varied enough to allow choices for classroom use and professional study. With so many topics considered, no one can be given extensive consideration. This is true even of geometric constructions, a topic which would seem to lend itself especially well to this type of treatment. However, this plan provides further evidence that *Secondary Mathematics* is not intended to replace other existing courses for teachers.

The style of writing is clear and direct. Examples are included of the various concepts or theorems developed. Practice exercises are sufficiently numerous

for advanced undergraduate or first year graduate students. There is a good list of references with each chapter. Answers are given for odd-numbered exercises. Many of the references to the secondary school come in the various chapters under the heading of Discussion. Illustrative of these would be:

"Compare the Euclidean solution of a locus theorem as presented here with that given in current high school textbooks. To what extent can the high school treatment be improved to include necessary and sufficient concepts? Should the analytic geometry method of locus be introduced in high school geometry? If so, to what extent?"

"The foregoing treatment of complex exponents is too difficult for high school presentation. At what point in collegiate mathematics should it first be presented? How can complex exponents be made meaningful to a senior high school mathematics class?"

As further illustration of the professionalized treatment of the material attention might be called to the list in the chapter on the derivative of problems on maxima and minima suitable for high school students.

College staff members who are interested in improving their offerings for teachers, particularly from the point of view of helping students in their classes appreciate the value for them, as secondary school teachers, of having a knowledge and understanding of mathematics beyond the calculus, should examine this book carefully. If all teachers of secondary mathematics had studied and understood the content of this book within college courses or as a part of their professional reading, surely more effective work would be done in our classrooms.

J. R. MAYOR
University of Wisconsin

The Algebra of Vectors and Matrices. By T. L. Wade. Cambridge, Mass., Addison-Wesley, 1951. ix+189 pages. $4.50.

According to its preface, this book is intended as "an elementary exposition of the *algebra of vectors and matrices*, that exposition being articulated with the basic concepts of modern algebra in the broad sense, to wit, group, integral domain, field, ring, basis, dimension and isomorphism." Also, the student is assumed to have progressed no further than analytic geometry and to know the elements of determinant theory.

The book may be recommended as an apparently usable one in a field which suffers from a paucity of texts, but not without a number of rather serious reservations.

First of all, one may question whether a student with only the stated minimal background is prepared for the book's immediate plunge into abstract group theory, and also whether he knows, without previous discussion, the meaning of such terms as, for example (p. 18), "one-to-one correspondence."

Secondly, one may cite a large number of slips of various types. For example, even and odd permutations are defined (p. 6), but the essential fact that a permutation cannot be both even and odd is not proved, nor is it even stated;

in defining a ring (p. 12), the right distributive law has been omitted; often (*e.g.*, p. 19) constant k lacks a necessary restriction: "$k \neq 0$." To this list may be added certain assertions which might perhaps just as well be deleted from the text. For example: (p. 48) "One should draw the distinction between 'perceptual' or visualizable spaces and 'conceptual' or logically conceived spaces. . . . Ordinary three space is usually thought of as a perceptual space, although a configuration in it of moderate complexity may not be altogether perceptual"; (p. 47) " . . . by a space of two dimensions is meant any set of objects which may be put in a one-to-one correspondence with the totality of vectors of two dimensions"; (p. 49) " . . . if a set of vectors are linearly dependent, their corresponding scalar coordinates are linearly dependent."

I. H. Rose
University of Massachusetts

An Introduction to Probability Theory and Its Applications. By William Feller. New York, John Wiley and Sons, Inc., 1950. xii+419 pages. $6.00.

The discussions and definitions of mathematical probability found in college algebra books reduce the theory to a pure *a priori* system and thereby detract from the real purpose of probability theory. Too often the problems and examples are either trivial or so detached from application as to appear ridiculous. The history of mathematics, as historians of science have shown, displays a fascinating interplay between theory and application. Certainly it is reasonable to expect that learning will be facilitated if some similar pattern is followed in the presentation of material to the neophyte.

Dr. Feller in this new text has such a novel view and presentation. As it should be with all basic concepts, he introduces the concept of probability with great caution by using intuitive examples before proceeding to a more rigorous and abstract definition. The use of sample space, in this volume discrete sample space, as the basis for definition is intuitively more pleasant and to this reader more elegant. Not only does this approach avoid the falseness of the usual ratio definition but it lends itself more readily to transition to statistical applications.

The symbolism, $Pr(A)$, for the probability of an event A is totally differentiated from the usual permutation symbol, which Dr. Feller carefully avoids. The symbol, $(n)_r$, is used to mean the permutation of n things r at a time, although the concept of permutations is not directly associated with the symbol in the text. Specifically the author states, "The number of different possible samples of size r from a population of n elements is n^r if the sampling is with replacement, and $(n)_r$ if it is without replacement." This may be hedging, but the approach seems to be more gratifying to this reader. Similarly the usual combination idea is introduced through partitions, the symbol used is $\binom{n}{r}$ for the number of different groups of size $r \leq n$ that can be selected from n elements. This treatment leads easily to multinomial coefficients.

The two chapters on combination of events and conditional probabilities

are as fine as ever have been written. The discussions of applications in these chapters are reason enough for merit. Two other features of these chapters that should not escape mention are the handling of the Laplace "law of succession" and the unique approach to conditional probability from definition thus avoiding a rote usage of Bayes' Rule.

Much of the material presented is of course standard; binomial and Poisson distributions, the normal distribution and the law of large numbers. However, the theory of recurrent events has been specially designed for this volume and leads directly to Markov Chains.

The various applications to occupancy problems, runs, trunking problems, waiting time, matching and guessing, *et cetera*, throughout the book should give the volume appeal to a diverse audience.

The book is not without error but, as might be expected, these are relatively rare. The definitions of event, page 13, and sample space, page 10, seem to be equivalent yet apparently not intended to be so, unless we conceive of the event as being the "act" that brings into being the sample space. If this be the intention of the author he has gone to great lengths to achieve it. Without being picayune it might also be pointed out that the concept of asymptotic equality, first introduced in Stirling's Formula, is vague. Still further a juxtaposition of definition and usage occurs concerning the concept of density function. The "bad" features of the book are, however, trivial by comparison and in no way detract from the readability or teachability of the work.

In general the text is more appealing than most probability texts. That it is a genuine contribution to the literature of mathematical statistics will be borne out by the tests of time and usage.

F. A. C. SÉVIER
Princeton, New Jersey

NEW BOOKS RECEIVED

Finite Deformation of an Elastic Solid. By F. D. Murnaghan. New York, John Wiley and Sons, Inc., 1951. 140 pages. $4.00.

Les Recreations Mathematiques (Parmi Les Nombres Curieux). By Victor Thébault. Paris, Gauthier-Villars, 1952, vi+297 pages. $7.50.

Introduction to Mathematical Thinking. By Friedrich Waisman. New York, Frederick Ungar Publishing Company, 1951. 12+260 pages. $4.50.

Elements of the Topology of Plane Sets of Points, Second Edition. By M. H. A. Newman. London, Cambridge University Press, 1951. vi+214 pages. 27s. 6d.

An Introduction to Applied Mathematics. By J. C. Jaeger. New York, Oxford University Press, 1952. xiii+445 pages. $7.00.

Aristotle's Syllogistic. By Jan Lukasiewicz. New York, Oxford University Press, 1951. xi+141 pages. $3.00.

Analytic Geometry. By W. K. Morrill. Scranton, International Textbook Company, 1951. 14+383 pages. $3.50.

Proceedings of a Second Symposium on Large-Scale Digital Calculating Machines. By Harvard University Computation Laboratory and the Navy Department Bureau of Ordnance. Cambridge, Harvard University Press, 1951. 38+393 pages. $8.00.

Tables of the Error Function and of Its First Twenty Derivatives (Annals 23). By the Staff of the Harvard University Computation Laboratory. Cambridge, Harvard University Press, 1952. 28+276 pages. $8.00.

Meaningful Mathematics. By H. S. Kaltenborn. New York, Prentice-Hall, Inc., 1951. xiv+397 pages. $4.75.

History of the Theories of Aether and Electricity. By Sir Edmund Whittaker. New York, The Philosophical Library, 1951. xiv+343 pages. $12.00.

Linear Transformation in n-Dimensional Vector Space. By H. L. Hamburger and M. E. Grimshaw. New York, Cambridge University Press, 1951. 10+195 pages. $4.50.

L'Enseignement des Mathematiques Generales Par Les Problemes, I. By G. Bouligand and J. Rivaud. Paris, Librairie Vuibert. 6+372+4 pages. 2000 francs.

Application of Linear Programming to the Theory of the Firm. By Robert Dorfman. Berkeley, University of California Press, 1951. ix+98 pages. $3.50.

Graphic Aids in Engineering Computation. By R. P. Hoelscher, J. N. Arnold, and S. H. Pierce. New York, McGraw-Hill Book Company, 1952. 8+197 pages. $4.50.

Trigonometry—Plane and Spherical. By L. L. Smail. New York, McGraw-Hill Book Company, 1952. xii+406 pages. $3.75.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of special features, student papers, bibliographies of program topics, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

CLUB REPORTS, 1950–51

Mathematics Club, Sacramento State College

The *Mathematics Club of Sacramento State College* was organized in January, 1951, and met monthly during the spring semester of 1951. The following topics were presented:

Logic of mathematics, by Dr. Hal D. Draper

Methods of integration: by parts and long division, by Dr. Louis C. Graue

Mathematical theory of games, by Mr. Norman Rudy

Development of number system, by Mr. W. W. Wallace, Sacramento Junior College

What is the difference between mathematics, philosophy of mathematics and philosophy? by Mr. Maxon A. Bernoff, Sacramento Junior College.

The officers elected for 1951–52 are: President, Robert Rimmele; Vice-President, George May; Secretary, John Otter; Treasurer, Mathias Hilling; Program Chairman, Robert Gustafson.

Mathematical Section of the Order of Magnitude, Pasadena City College

The *Mathematical Section of the Order of Magnitude* meets every two weeks at *Pasadena City College.* The topics presented in 1950–51 were:

Mathematical models and soap film experiments, by Mr. W. H. Glenn

Vector analysis, by Robert Phelan

Congruences of numbers, by Marvin Ridley

The seven bridges of Königsberg, by Stuart Bowen

Projective geometry, by Mr. Laing

Relationships and correspondences between slopes and their eccentricities, by Mr. Lewis

Fourier series, by Mr. Hardy

Mathematics conference report, by Marvin Ridley

Prime numbers, by Marvin Ridley

Hyperbolic trigonometric functions, by William Lindley

Higher plane curves, by Mr. Hardy

Logic in mathematics, or What is mathematics? by Dr. E. T. Bell.

Both the *Mathematical Section* and the *Astronomy Section of the Order of Magnitude* participated in Club Day and Vocational Guidance Day at the College.

Officers for the second semester were: President, Andy Markell; Vice-President, Bob Phelan; Secretary-Treasurer, Carol Blackinton.

Pi Mu Epsilon, University of Oregon

The *Oregon Alpha* chapter of *Pi Mu Epsilon* held four regular meetings including initiations and elections during 1950–51. The following papers were read:

Scales of notation, by Mr. Wesley L. Nicholson

A problem in symbolic logic, by Miss Dorothy J. Christensen.

The DeCou Prize award of $50 given annually to the outstanding upper division student in mathematics was presented to Miss Dorothy J. Christensen.

The annual joint spring picnic of the chapter and the Mathematics Department of the University of Oregon was held May 15.

Officers elected for 1951–52 are: Director, Wesley L. Nicholson; Vice-Director, Roger H. Moore; Secretary-Treasurer, Sam Saunders.

Junior Mathematics Club, University of Chicago

The *Junior Mathematics Club of the University of Chicago* meets bi-weekly, its main activities being the talks presented at these meetings and the semi-

weekly teas given for the Department of Mathematics. During 1950–51 the following talks were presented:

Notion of mathematical structure, by Donald Bratton
Lie-admissible algebras, by Louis Weiner
Structure theory of algebras and rings, by Alexander Rosenberg
Integral inequalities suggested by game theory, by Joseph Bram
The weak law of large numbers, by Dr. Murray Rosenblatt
Kolchin: Picard-Vessiot theory, by William Howard
Tate: functional equation of the zeta function, by Sanford Colbert
Method of gradients for the determination of the characteristic roots and vectors of a symmetric matrix, by Dr. William Karush
Weak topologies, by Ernest Michael
Curves of constant breadth, by William Ballard
Some integral inequalities, by Daniel Waterman
Partitions of unity and Tietze's extension theorem, by Robert Bartle.

The activities of the club were highlighted by two parties and the inauguration of the club reading room.

Officers elected for 1951–52 are: President, Fred Wright; Treasurer, Thomas King.

Kappa Mu Epsilon, Kansas State Teachers College, Emporia

The *Kansas Beta* chapter of *Kappa Mu Epsilon* at *Emporia State Teachers College* met monthly throughout the school year 1950–51. Papers presented at these meetings were:

The 200-inch telescope, by Brooks Becker
Short cuts in arithmetic, by George Crumley
Mechanical brains, by Richard Shur
Electronic computers, by Robert Klotz
Similar triangles, by Elva Libben
History of negative numbers, by Richard McAlister
History and development of Kappa Mu Epsilon, by Dr. O. J. Peterson
Galaxies and nebulae, by Raymond Eccles
Structure of the universe, by Billy Burgert
Famous calculus problems, by Don Allison
Probability, by Dale Smith
How to solve it, by A. H. Albert
Report on present occupations of former Kappa Mu Epsilon members, by Don Hanifan
Report on meeting of Kansas Association of Teachers of Mathematics, by Max Woods and Richard Gregory
Report on the national convention of Kappa Mu Epsilon at Springfield, Missouri, by the eleven delegates.

Delta-Y Club, D'Youville College

Delta-Y members and guest speakers this year discussed many of the interesting side-lights of mathematics. Each monthly meeting included one or more of the following topics:

An introduction to topology, by Miss Helen Hand
Mathematics and the business world, by Miss Alice Staebell
Some envelopes of the Lemoine line, by Miss June McArtney
Poetry and mathematics, by Helen Bertell
Magic squares, by Emily Monczynski
Mathematical wrinkles, by Shirley Hebler
Astronomy with an opera glass, by Ann Dolce
Mathematics and the imagination, by Marian Adrian
The fourth dimension, by Mary Frances McHugh
Panel discussion of the sliderule, by Mary Catherine Forde, Patricia Leising, Shirley Hebler, and Emily Monczynski.

The President elected for 1951–52 is Marilyn Repsher.

Mathematics Club, Marquette University

The following lectures were presented to the *Mathematics Club of Marquette University* during 1950–51:

Trisection of the angle, by David Rux
Squaring the circle, by Robert Wurm
Mathematical puzzles, by Dorothy Stidola
Mathematics and music, by Keith Betz
Magic squares, by Frank Michuden
Flatland, by Irving Treu
Unclear physics, by Dr. Barkow.

The President elected for 1951–52 is Philip Blank.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

PURCHASE OF THE A.M.S. LIBRARY BY THE UNIVERSITY OF GEORGIA

The American Mathematical Society has sold its Library to the University of Georgia. This library consists of between thirteen and fourteen thousand bound volumes. These added to the not inconsiderable collection already owned by the University gives the University of Georgia a very fine library for research

in mathematics. It is particularly strong in long runs of periodicals. Its collection of doctor's dissertations, not only American but foreign, is unrivaled. The University of Georgia is continuing the entire exchange list of the American Mathematical Society and expects to do all that is possible to keep the dissertation collection up to date. It is believed that a complete file of dissertations can be administered for the benefit of mathematicians everywhere. With this in mind department chairmen are urged to report the names of all persons who take the Ph.D. degree to the librarian of the University of Georgia at Athens. He will contact the author relative to a copy of the dissertation.

It is hoped that all mathematicians will feel free to utilize any item in the entire collection. It is the desire of the University of Georgia to organize its library for the benefit of all competent users, and rules for administering the mathematics collection are being framed with this in mind. It is particularly hoped that mathematicians in the Southeast will be benefited by the location of this library at Athens.

PRELIMINARY ACTUARIAL EXAMINATIONS AND PRIZE AWARDS

The Society of Actuaries has announced the winners of the prize awards for the May 1951 Preliminary Examinations. These awards were given to the nine undergraduates who ranked highest in score for Part 2 of these examinations. The first prize of $200 was awarded to F. O. Wyse of Harvard University. Additional prizes of $100 each were awarded to the following: A. R. Auer, Columbia University; Anita Bredt, University of Pennsylvania; W. E. Chapman, III, Duke University; J. P. Lazarus, Rensselaer Polytechnic Institute; Judith A. Richman, University of Pennsylvania; J. B. Ross, Harvard University; C. D. Silletto, University of Iowa; Howard Young, City College of the City of New York.

On May 21, 1952, the Preliminary Actuarial Examinations will be given to undergraduate students of mathematics who may be interested in going into the actuarial profession. These examinations consist of three parts: (1) Language Aptitude Examination, (2) General Mathematics Examination, and (3) Special Mathematics Examination. For detailed information regarding these examinations write to: The Society of Actuaries, 208 South La Salle Street, Chicago 4, Illinois.

The Society of Actuaries will award one $200 and eight $100 prizes to the nine successful undergraduates who rank highest in the General Mathematics Examination and who concurrently pass, or previously have passed, the Language Aptitude Examination.

ESSAY CONTEST OF THE INSTITUTE FOR THE UNITY OF SCIENCE

The Institute for the Unity of Science is conducting an Essay Contest during 1952. The theme of the Contest is Mathematical Logic as a Tool of Analysis. The essay should contain a detailed account of the actual accomplishments which may be credited to the techniques of modern formal logic in both applied

and theoretical science as well as in philosophy. On the other hand, considerations of the role of these techniques in researches on the foundations of pure mathematics and of formal logic are specifically excluded from the scope of the essay. But a discussion of unsuccessful or dubious applications of mathematical logic is highly pertinent. Proposed extensions of these techniques to new problems and areas, or critiques of the limitations of these techniques, should be developed in terms as concrete as possible.

A prize of $500 will be awarded for the best essay submitted on the theme, and two additional prizes of $200 each will be given for the two next best essays. For further information regarding rules of the Contest write to: Institute for the Unity of Science, American Academy of Arts and Sciences, 28 Newbury Street, Boston 16, Massachusetts.

REQUEST FOR BOOKS

The Franklin and Marshall College Library wishes to purchase the following: Watson—Complex Integration and Cauchy's Theorem; Eagle—Fourier's Theorem; Emch—Projective Geometry; Schmall—Analytic Geometry, 2nd edition; Scioberretie—mathematics book published about 1933, or later; Silberstein—Synopsis of Applicable Mathematics. Anyone who wishes to dispose of these or other advanced mathematics books and periodicals may write to Franklin and Marshall College Library, Lancaster, Pennsylvania.

SUMMER COURSES

The following institutions announced advanced courses in mathematics for the summer of 1952:

The Catholic University of America. June 30 to August 9: Professor Ramler, college geometry, synthetic projective geometry, ordinary differential equations; Professor Finan, theory of equations II; Professor Rice, advanced calculus I, calculus of observations; Mr. Clark, partial differential equations of mathematical physics.

Columbia University, Department of Mathematics. July 7 to August 15: Professor Koopman, introduction to higher algebra, Fourier series and integrals; Professor Rosenbloom, differential equations, mathematical logic; Professor Murray, probability, numerical methods for the integration of ordinary and partial differential equations; Professor Lorch, foundations of mathematics, theory of functions of a complex variable; Professor Levi, differential geometry; Professor Kolchin, ideal theory in commutative rings.

Columbia University, Teachers College. July 7 to August 15: Professor Fehr, teaching arithmetic in the elementary school, professionalized subject matter in advanced secondary mathematics, part II; Professor Rosskopf, teaching non-academic mathematics in the high school, supervision and problems of teaching secondary school mathematics; Professor Shuster, business arithmetic and mathematics, field work in mathematics; Professor Van Engen, history of mathematics, teaching algebra in secondary schools; Professor Yates, applications of plane geometry, survey of higher mathematics for teachers.

De Paul University. June 9 to August 5: Professor De Cicco, calculus of variations, differential geometry; Professor Caton, statistical sampling theory, differential equations.

Duke University. June 11 to July 19: Professor Carlitz, finite differences, thesis seminar; Professor Thomas, differential equations, projective geometry. July 22 to August 29: Professor Dressel, solid analytic geometry; Professor Gergen, mathematical statistics. August 5 to August 15: Professor W. W. Rankin, Director, Institute for Teachers of Mathematics.

Kent State University. June 16 to July 25: Professor Johnson, college geometry; Professor Olson, differential geometry; Professor Brooks, mathematical statistics. July 28 to August 29: Professor Kaiser, differential equations, probability.

Massachusetts Institute of Technology. June 9 to July 18: Professor Tucker, Princeton University, theory of games. July 21 to August 29: Professor Rademacher, University of Pennsylvania, special functions.

Michigan State College. June 23 to August 1: Professor Grove, differential geometry; Professor Hill, complex variable; Professor Powell, advanced calculus I; Professor Stewart, college geometry, theory of numbers; Mr. Coy, correlation analysis. June 22 to August 22: Professor Bell, theory of matrices and groups, higher algebra; Professor Harter, analysis of variance; Professor Herzog, Fourier series, advanced calculus II; Professor Kelly, projective geometry; Professor Nordhaus, theory of equations; Professor Stelson, differential equations; Professor Wells, vector analysis, integral equations.

State University of Iowa. June 9 to August 6: Professor Chittenden, matrices and determinants, infinite series; Professor Conkwright, differential equations; Professor Hogg, elements of statistics, statistical hypotheses; Professor Knowler, mathematics of business and industry, topics in actuarial science; Professor Smiley, introduction to advanced mathematics, Galois theory of fields; Professor Wylie, astronomy.

University of Buffalo. June 30 to August 9: Professor Gehman, synthetic projective geometry; Professor Montague, introduction to topics in higher algebra, history of mathematics; Professor Schneckenburger, differential equations, theory of sets.

University of California at Los Angeles. June 23 to August 2: Professor Van der Corput, University of Amsterdam and Institute for Numerical Analysis, asymptotic expansions; Professor Kaplansky, University of Chicago, higher algebra; Professor Arens, complex variables.

University of Chicago. June 23 to August 30: Professor Spanier, introduction to the theory of groups and rings, convexity; Professor Chern, fundamental concepts of geometry, differential geometry II; Professor Stone, theory of sets; Professor Barnard, tensor analysis; Professor Halmos, theory of functions of real variables, topological algebra II; Professor Segal, Lebesgue and Stieltjes integrals, mathematical foundations of quantum mechanics; Professor Schilling, representation of finite groups.

University of Colorado. June 16 to August 26: Professor Scherk, University of Saskatchewan, differential geometry; Professor Edrei, differential equations, functions of a real variable; Professor Jones and Professor Kendall, modern geometry; Professor Snively, vector analysis; Professor Stahl, Fourier series and boundary value problems. July 24 to August 26: Professor Brown, University of Tennessee, teaching of secondary mathematics, mathematics workshop in curriculum problems.

University of Kentucky. June 16 to August 9: Professor Cowling, tensor analysis; Professor Goodman, vector analysis; Professor Pence, college geometry; Professor South, statistics seminar; Professor Ward, theory of groups.

University of Michigan. June 23 to August 15: Professor Anning, college geometry; Professor Brauer, higher algebra, structure of rings; Professor Carver, theory of statistics; Professor Churchill, heat conduction and diffusion; Professor Coburn, operational mathematics, vector analysis; Professor Craig, analytic theory of frequency functions; Professor Dwyer, multivariate analysis, probability; Professor Hay, theoretical mechanics, tensor analysis; Professor E. H. C. Hildebrandt, history of geometry, teaching of algebra; Professor Kaplan, functions of complex variable and applications, partial differential equations; Dr. Leisenring, synthetic projective geometry; Professor Nesbitt, intermediate mathematics of life insurance; Professor Piranian, infinite series and summability; Professor Rainich, quadratic numbers and quadratic forms, potential theory; Professor Rainville, intermediate differential equations; Professor Samelson, theory of functions of a complex variable, differential geometry; Professor Tornheim, theory of matrices; Professor Young, foundations of mathematics, unified topology.

University of Minnesota, Department of Mathematics. June 17 to July 26: Professor Gelbaum, advanced calculus I, Fourier series and orthogonal functions; Professor Loud, theory of numbers, calculus of finite differences. July 29 to August 30: Professor Gelbaum, advanced calculus II, special functions; Professor Hatfield, advanced algebraic theory, solid analytic geometry.

University of Minnesota, Institute of Technology. June 16 to July 25: Professor Loye, differential equations, intermediate calculus; Professor Munro, vector analysis, theory of complex variables. July 28 to August 29: Professor Turrittin, advanced calculus, vector analysis with applications.

University of North Carolina. June 12 to July 16: Professor Winsor, college geometry; Professor Garner, calculus of finite differences; Professor Hill, elementary mathematical statistics; Professor Cameron, vector spaces and matrices; Professor Hoyle, advanced calculus I; Professor Brauer, some recent results in algebra. July 17 to August 20: Professor Linker, differential equations; Professor Lasley, analytic projective geometry; Professor Jones, elementary point set theory; Professor Whyburn, advanced calculus II.

University of Oklahoma. June 6 to August 1: Professor Bernhart, college geometry; Professor Springer, elementary differential equations; Professor LaFon, ordinary and partial differential equations; Professor Andree, theory of

numbers; Professor Brixey, theory of groups; Professor Goffman, combinatorial topology.

University of Southern California. June 23 to August 1: Professor Snapper, advanced calculus, theory of numbers; Professor Henkin, foundations of mathematics; Professor Yood, calculus of variations, advanced topics in analysis.

University of Washington. June 23 to July 23, July 24 to August 22: Professors Brownell, Haller, and Leipnik, ordinary and partial differential equations; Professor Avann, vector analysis; Professor Paulson, matrices and determinants; Professor Yagi, higher calculus; Professor Ball, introduction to modern algebra; Professor McFarlan, calculus of variations.

University of Wisconsin. June 30 to August 22: Professor Bing, topics in geometry, higher mathematics for engineers; Professor Buck, survey of the foundations of arithmetic; Professor Fullerton, advanced analytic geometry; Dr. Immel, introduction to statistics; Professor Sokolnikoff, higher mathematics for engineers, determinants and matrices; Professor Young, Schwartz' theory of currents and distributions; Professor Mann, Ohio State University, topics in mathematical statistics, topics in the theory of numbers.

University of Wyoming. June 9 to July 11: Professor Varineau, theory of equations, fundamental concepts of mathematics; Professor Schwid, ordinary differential equations, solid analytic geometry; Professor Neubauer, history of mathematics; Professor W. N. Smith, mathematical theory of probability. July 14 to August 15: Professor Barr, advanced calculus; Professor S. R. Smith, partial differential equations; Professor Steen, college geometry.

West Virginia University. June 4 to July 15: Professor Peters, differential equations, group theory; Professor Stewart, theory of probability. July 16 to August 22: Professor Cunningham, modern geometry, theory of determinants and analytic geometry of space; Professor Vehse, calculus of variations.

ANNUAL CONFERENCE FOR TEACHERS OF MATHEMATICS

The Second Annual Conference for Teachers of Mathematics will be held from June 30 to July 12 on the Los Angeles campus of the University of California, according to Clifford Bell, Head of Mathematics Extension for the University.

Purpose of the conference is to bring together teachers interested in mathematics to study common problems in the teaching of mathematics and to learn new uses of mathematics in various fields of endeavor.

The University's Department of Mathematics, Education and Mathematics Extension, and University Extension are sponsoring the conference in cooperation with the California Mathematics Council and the National Council of Teachers of Mathematics. W. W. Rankin, Director and Professor of Mathematics, Duke University, will come to Los Angeles to preside at the conference. Members of the planning committee are L. J. Adams, chairman, department of mathematics, Santa Monica City College; Dale Carpenter, academic education branch, Curriculum Division of the Los Angeles City Schools; Reuben Palm,

director, secondary education, Los Angeles County Schools and Paul White, Head, department of mathematics, University of Southern California.

Information concerning registration and living accommodations is available on request to the Department of Conferences, University of California Extension, Los Angeles 24.

PERSONAL ITEMS

Professor H. M. Gehman, Secretary-Treasurer of the Association, was the representative of the Association at the conference of organizational members of the American Council on Education which was held in Washington, D. C. on January 25–26, 1952.

Professor Einar Hille was elected to the Royal Swedish Academy of Sciences on April 11, 1951.

Professor Anne L. Lewis of the Woman's College of the University of North Carolina represented the Association at the inauguration of President C. C. Jernigan of Queens College, Charlotte, North Carolina on March 29, 1952.

Professor W. T. Martin of the Massachusetts Institute of Technology represented the Association at the inauguration of President L. W. Jones of Rutgers University.

Frank B. Jewett Postdoctoral Fellowships of the Bell Telephone Laboratories for 1952–1953 have been awarded to the following mathematicians: Mr. Murray Gerstenhaber of Harvard University, Dr. Henry Helson of Yale University, and Mr. E. T. Onat of Brown University.

Arizona State College announces the following: Mr. Joseph Kindle, formerly of the University of Cincinnati, has been appointed Special Lecturer for the year 1951–52; Professor F. C. Gentry is on leave for the year 1951–52 and is teaching at the University of New Mexico.

California Institute of Technology announces the following appointments for 1952–1953: Dr. H. A. Dye, former Bateman Fellow, has been promoted to an instructorship; Mr. F. B. Fuller of Princeton University has been appointed to an Instructorship; Mr. R. A. Dean of Ohio State University has been appointed Bateman Fellow.

Case Institute of Technology announces that Dr. R. A. Clark and Dr. F. C. Leone have been promoted to assistant professorships.

Hampton Institute reports the following: Professor R. D. Gordon of Washington College has been appointed to an associate professorship; Miss Gloria C. Ford, previously a student at the University of Pennsylvania, has been appointed to an instructorship; Miss Bobbie E. Jones, formerly a graduate fellow at Howard University, has been appointed to an instructorship; Miss Rosalind M. Eagleson and Miss Elaine H. Young are on leave of absence.

Illinois Institute of Technology announces that Mr. W. F. Darsow and Mr. Albert Soglin have been appointed to instructorships. On January 10, 1952 the Mathematics Department of the Institute opened an exhibition on Geometry in the Museum of Science and Industry, Chicago; a guidebook explaining the exhibition was published by Professor Karl Menger.

Iowa State College makes the following announcements: Assistant Professor N. P. Yeardley of Purdue University has been appointed to an assistant professorship; Mr. W. J. Schwartz, formerly an instructor at Montana State College, has been appointed to an instructorship; Graduate Assistants Karl Conrad, R. W. Klopfenstein and H. D. Mills have been promoted to instructorships.

Miami University reports that Mr. C. C. Crell and Mr. E. R. Epperson, previously graduate students at the University, have been appointed to instructorships.

Michigan State College reports the following: Associate Professor C. P. Wells has been promoted to a professorship; Assistant Professor E. A. Nordhaus has been promoted to an associate professorship; Dr. Ingram Olkin, formerly research associate at the University of North Carolina, has been appointed to an assistant professorship; Dr. J. B. Kelly, who was a member of the Institute for Advanced Study during 1950–51, has been appointed to an instructorship; Mr. K. H. Carlson, formerly graduate assistant at the University of Wisconsin, and Mr. J. G. Hocking, previously teaching fellow at the University of Michigan, have been appointed to instructorships.

Mount Holyoke College reports: Professor Marie Litzinger is on leave of absence; Assistant Professor I. H. Rose of the University of Massachusetts has been appointed to a visiting professorship.

Northwestern University announces the following: Instructor W. M. Boothby has been promoted to an assistant professorship; Dr. George Springer, previously Moore instructor at Massachusetts Institute of Technology, has been appointed to an assistant professorship; Dr. J. S. MacNerney, Mr. W. E. Jenner, and Mr. T. C. Littlejohn have been appointed to instructorships; Dr. Oswald Wyler has been appointed Lecturer; Professor R. P. Boas is on leave for the year on a Guggenheim Fellowship.

Oregon State College makes the following announcements: Assistant Professors B. H. Arnold and B. W. Brewer have been promoted to associate professorships; Professor W. E. Milne has returned to the headship of the department after a year's leave of absence during which he was consultant at the National Bureau of Standards, Institute for Numerical Analysis at Los Angeles; Dr. F. E. Ehlers has been granted leave for 1951–52 in order to accept a position with the Aerodynamics Unit of the Engineering Department of Boeing Aircraft in Seattle. Dr. H. W. Eves was the recipient of the Carter Award for 1950–51; this award is given to "the best teacher in the sciences" according to student vote.

The University of Delaware announces the appointments of Dr. J. H. Barrett to an assistant professorship and Mr. A. C. Nelson to an instructorship.

At the University of Georgia: Professor H. S. M. Coxeter of the University of Toronto gave the Colloquium Lectures entitled "Regular Honeycombs in Elliptic Space" on January 8–11, 1952; he also lectured under the auspices of Pi Mu Epsilon on "The Golden Section."

University of Hawaii reports: Assistant Professor Christopher Gregory has been promoted to an associate professorship; Dr. Paolo Comba, formerly a student at the California Institute of Technology, has been appointed to an assistant professorship.

University of Massachusetts announces the appointments of Mr. Bernard Bussel and Mr. A. A. Kheiralla to instructorships and of Mr. Sidney Kissin to a graduate assistantship.

At Yale University: Assistant Professor C. E. Rickart has been promoted to an associate professorship; Associate Professor G. P. Hochschild of the University of Illinois has been appointed Visiting Associate Professor for the year 1951–52; Dr. Henry Helson, who was a lecturer at the University of Uppsala, Sweden during the year 1950–51, has been appointed to an instructorship; Dr. John Wermer, previously a teaching fellow at Harvard, has been appointed to an instructorship; Mr. J. Y. Barry, Mr. G. R. Dubé, Mr. L. W. Green, Mr. R. B. Smith and Mr. Morris Weisfeld have been appointed Assistants-in-Instruction for the year 1951–52; Professor Nathan Jacobson is on leave of absence during the year 1951–52 and is spending the year in France on a Guggenheim Fellowship and Fulbright Award.

Yeshiva University announces: Assistant Professor Henry Lisman has been promoted to an associate professorship; Mr. Barry Ginsburg has been appointed to an assistant instructorship.

Associate Professor R. E. Bellman of Stanford University is on leave of absence and has accepted a temporary position at the Forrestal Research Center, Princeton University.

Mr. Jonas Beraru has a position as Design Engineer for the Ford Instrument Company, Long Island City, New York.

Mr. R. W. Bird has accepted a position as Statistical Accountant with the Pacific Indemnity Company, Los Angeles, California.

Dr. A. W. Boldyreff, formerly a staff member of Sandia Corporation, Albuquerque, New Mexico, is now a research specialist at North American Aviation, Downey, California.

Mr. T. C. Kimball, mathematics master at the Lawrenceville School, died on January 8, 1952. He had been a member of the Association for twenty-five years.

Reverend G. A. O'Donnell of Boston College died on January 1, 1952. He had been a member of the Association for twenty-seven years.

Professor E. W. Schreiber, head of the Mathematics Department, Western Illinois State Teachers College, died on December 17, 1951. He had been a member of the Association for thirty-one years.

Dr. P. F. Neményi of the Naval Research Laboratory, Washington, D. C. died on March 1, 1952.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following ninety-six persons have been elected to membership by the Board of Governors on applications duly certified.

A. G. ANDERSON, Ph.D. (Michigan) Instr., Oberlin College, Ohio

R. R. ARCHER, Student, Massachusetts Institute of Technology, Cambridge, Mass.

W. A. BECK, B.S. (Case) Instr., Purdue University, Lafayette, Ind.

J. E. BOSSHART, M.A. (Northwestern) Lecturer, Western Reserve University, Cleveland, Ohio

JENEVA J. BREWER, M.A. (Wichita) Instr., University of Wichita, Kans.

A. T. BUTSON, M.A. (Michigan S. C.) Grad. Assistant, Michigan State College, East Lansing, Mich.

J. F. CANU, M.Sc. (Chicago) Mathematician, U. S. Naval Proving Ground, Dahlgren, Va.

F. D. CHAMBERLAIN, B.A. (Western Washington) Capt., U. S. Air Force, Hamilton, Calif.

N. A. CHILDRESS, M.A. (Mississippi) Instr., University of Mississippi, University, Miss.

K. P. CLANCY, Student, Fordham University, New York, N. Y.

PAOLO COMBA, Ph.D. (C.I.T.) Asst. Professor, University of Hawaii, Honolulu, Hawaii

S. D. CONTE, Ph.D. (Michigan) Asst. Professor, Wayne University, Detroit, Mich.

D. W. CROWE, M.A. (Nebraska) Teaching Fellow, University of Michigan, Ann Arbor, Mich.

REV. F. T. DALY, M.S. (St. Louis) University of Notre Dame, Ind.

C. R. DEPRIMA, Ph.D. (N.Y.U.) Head, Mathematics Branch, Office of Naval Research, Washington, D. C.

W. A. DOLID, Student, Alfred University, N. Y.

JIM DOUGLAS, JR., M.A. (Rice) Asst., Rice Institute, Houston, Tex.

LEILA A. DRAGONETTE, Ph.D. (Pennsylvania) Research Associate, University of Chicago, Ill.

M. E. DRUMMOND, JR., B.S. (East Central S. C.) Research Assistant, University of Oklahoma, Norman, Okla.

J. E. F. DUTT, B.S. (Manhattan) Grad. Student, Columbia University, New York, N. Y.

R. A. EBNER, Ed.M. (Harvard) Instr., University of Connecticut, Storrs, Conn.

J. G. ELLIOTT, M.A. (Michigan S. C.) Grad. Assistant, Michigan State College, East Lansing, Mich.

D. E. ENOS, Student, Baker University, Baldwin City, Kans.

TREVOR EVANS, M.A. (Oxford) Asst. Professor, Emory University, Ga.

J. E. FORBES, M.S. (Bradley) Instr., Bradley University, Peoria, Ill.

MRS. VIRGINIA FORBES, M.S. (Illinois) Teacher, Community Unit High School, Heyworth, Ill.

A. O. FOREHAND, JR., B.S. (Rollins) Grad. Assistant, University of Alabama, Tuscaloosa, Ala.

R. R. FOSSUM, B.S. (Idaho) Ensign, U. S. Navy, San Francisco, Calif.

BARRY GINSBURG, B.A. (Yeshiva) Asst., Yeshiva University, New York, N. Y.

HERBERT GOERTZEL, A.B. (N.Y.U.) Mathematics Assistant, Oak Ridge National Laboratory, Tenn.

SAMUEL GOLDBERG, Ph.D. (Cornell) Instr., Lehigh University, Bethlehem, Pa.

D. E. GOWAN, B.A. (South Dakota) Computer, General Electric Company, Richland, Wash.

FLORENCE N. GREVILLE, M.A. (Columbia) Analyst, Department of Defense, Washington, D. C.

L. A. GUEST, M.S. (Oklahoma A & M) Grad. Assistant, University of Oklahoma, Norman, Okla.

LAURA GUGGENBUHL, Ph.D. (Bryn Mawr) Asst. Professor, Hunter College, New York, N. Y.

R. L. GULLEY, B.A. (Roosevelt) 2121 F Street, N. W., Washington, D. C.

B. F. HANDY, JR., M.S. (Howard) Research Assistant, Massachusetts Institute of Technology, Cambridge, Mass.

J. D. HANKINS, M.S. (Missouri) Instr., University of Missouri, Columbia, Mo.

MRS. LOUISE F. HANSON, B.S.E.E. (Wisconsin) Asst. Professor, Olivet College, Mich.

ARTHUR HERSKOWITZ, Junior Accountant, Greenberg & Tucker, New York, N. Y.

R. F. HOELKER, Ph.D. (Münster) Redstone Arsenal, Huntsville, Ala.

J. E. HOFFMAN, B.S.C.E. (Oklahoma) Grad. Assistant, University of Oklahoma, Norman, Okla.

R. C. JAMES, Ph.D. (C.I.T.) Asso. Professor, Haverford College, Pa.

J. H. B. KEMPERMAN, Ph.D. (Amsterdam) Visiting Asst. Professor, Purdue University, Lafayette, Ind.

R. W. KLOPFENSTEIN, M.S. (Iowa S. C.) Instr., Iowa State College, Ames, Iowa

D. L. KUNTZ, B.A. (Knox) Illinois Bell Telephone Company, Rock Island, Ill.

L. S. LEVY, Student, Yeshiva University, New York, N. Y.

VICTOR LEWICKE, A.M. (Columbia) Mathematician, Army Map Service, Washington, D. C.

A. C. LINDBERG, M.A. (Nebraska) Asso. Professor, Dana College, Blair, Neb.

WALTER LITTMAN, Student, New York University, N. Y.

J. M. LONG, B.S. (William & Mary) Grad. Assistant, Michigan State College, East Lansing, Mich.

C. P. LOUTHAN, 78 West First Avenue, Columbus, Ohio

BECKHAM MARTIN, Ph.D. (Purdue) Physicist, Owens-Illinois Glass Company, Toledo, Ohio

SAMUEL MELAMED, B.S. (McGill) Lecturer, McGill University, Montreal, Quebec

W. E. MIENTKA, M.A. (Columbia) Instr., University of Massachusetts, Amherst, Mass.

JACK MINKER, M.S. (Wisconsin) Dynamics Engineer, Bell Aircraft Corporation, Niagara Falls, N. Y.

C. R. MITCHELL, M.E. (Rhode Island C. of Ed.) Asst. Professor, Rhode Island College of Education, Providence, R. I.

G. C. NOONAN, JR., U. S. Navy, San Diego, Calif.

E. S. NORTHAM, M.S. (Michigan) Grad. Assistant, Michigan State College, East Lansing, Mich.

ILSE L. NOVAK, Ph.D. (Radcliffe) Member, Institute for Advanced Study, Princeton, N. J.

R. D. ORR, Student, Baker University, Baldwin City, Kans.

E. N. PARKER, Ph.D. (C.I.T.) Instr., University of Utah, Salt Lake City, Utah

W. M. PEREL, M.A. (Indiana) Teaching Assistant, Indiana University, Bloomington, Ind.

L. M. RAUCH, Ph.D. (California) Research Engineer, University of Michigan, Ann Arbor, Mich.

HELENE RESCHOVSKY, Ph.D. (Vienna) Instr., University of Connecticut, Storrs, Conn.

P. B. RICHARDS, M.S.M.E. (Harvard) Analytical Engineer, Babcock & Wilcox Research Laboratory, Alliance, Ohio

DOLORES A. RICKENS, M.A. (Columbia) Asso. Professor, College of St. Elizabeth, Convent Station, N. J.

V. E. ROBERTS, B.S. (Massachusetts) Instr., University of Alaska, College, Alaska

LOUIS ROBINSON, M.A. (Syracuse) Grad. Student, Syracuse University, N. Y.

WELLINGTON ROUNDS, Student, Rutgers University, New Brunswick, N. J.

D. A. RUSSO, M.S. (N.Y.U.) Grad. Student, New York University, N. Y.

F. P. J. SANSOM, Student, Baker University, Baldwin City, Kans.

J. F. SCHELL, B.S. (Dayton) Grad. Assistant, Indiana University, Bloomington, Ind.

S. L. SCHLUNEGER, Student, University of Tulsa, Okla.

L. L. SCOTT, Ph.D. (Illinois) Asst. Professor, University of Mississippi, Miss.

MILTON SIEGEL, B.S. (C.C.N.Y.) Mathematician, U. S. Naval Proving Ground, Dahlgren, Va.

T. F. SINGLETON, B.S. (St. Louis) Instr., Regis College, Denver, Colo.

F. M. SIOSON, Weather Bureau, Geophysical Observatory, Quezon City, Philippines

SISTER M. XAVERIA, Ph.D. (Michigan) Professor, Marygrove College, Monroe Campus, Mich.

SISTER RITA, M.A. (Catholic) Head, Mathematics Department, Trinity Preparatory School, Ilchester, Md.

S. T. SMITH, Student, University of Rochester, N. Y.

GEORGE SPRINGER, Ph.D. (Harvard) Asst. Professor, Northwestern University, Evanston, Ill.

DORIS S. STOCKTON, M.Sc. (Brown) Instr., University of Connecticut, Storrs, Conn.

JOSEPH TAJEN, B.S. (C.C.N.Y.) Test Engineer, Ordnance Aerophysics Laboratory, Daingerfield, Tex.

O. E. TAULBEE, M.A. (Michigan S. C.) Grad. Assistant, Michigan State College, East Lansing, Mich.

PETER TERWEY, JR., M.A. (Texas) Grad. Student and Part-time Instr., University of North Carolina, Chapel Hill, N. C.

R. P. TEUTSCH, Student, Florida Southern College, Lakeland, Fla.

W. J. THOMSEN, M.S. (Iowa) Research Associate, University of Iowa, Iowa City, Iowa

D. B. J. TOMIUK, B.Sc. (Loyola, Montreal) McGill University, Montreal, Quebec

R. H. TSOW, Student, Gonzaga University, Spokane, Wash.

D. J. WALKER, M.A. (Nebraska) Asst. Professor, University of Idaho, Moscow, Idaho

CECILIA T. WELNA, M.A. (Connecticut) Asst. Instr., University of Connecticut, Storrs, Conn.

F. G. WERNER, Student, Miami University, Oxford, Ohio

E. T. WESTER, M.A. (Oklahoma) Asst. Professor, Central State College, Edmond, Okla.

E. S. WOLK, M.Sc. (Brown) Instr., University of Connecticut, Storrs, Conn.

HENRY ZATZKIS, Ph.D. (Syracuse) Instr., University of Connecticut, Storrs, Conn.

REPORT OF THE TREASURER FOR THE YEAR 1951

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1951. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

I. TOTAL FUNDS OF THE ASSOCIATION ON JANUARY 1, 1951

M & T Trust Co., Buffalo	$ 8,705.06	Current Fund	$ 8,705.06
Securities	59,952.08	Carus Fund	7,701.88
		Chace Fund	10,117.07
		Houck Fund	8,214.71
		Chauvenet Fund	626.05
		General Fund	33,292.37
	$68,657.14		$68,657.14

II. Current Fund

Balance, January 1, 1951	$ 8,705.06	Monthly	
Dues	14,742.70	Publication	$14,553.91
Initiation fees	564.00	Reprints (net)	260.68
Subscriptions	3,992.16	Editor's Office	986.28
Sale of back numbers	716.75	Secretary-Treasurer's Office	
Advertisements	3,744.00	Clerical help	6,568.56
Sale of exchange periodicals	69.25	Postage and printing	1,666.08
Interest on General Fund	1,150.76	Office expenses and furniture	646.37
Income from Hardy Fund	120.00	Bank fees	101.04
Transfer from General Fund	4,484.95	Auditing fee	50.00
		Board of Governors	1,674.59
		Meetings	686.17
		Representatives	222.42
		Subventions	988.76
		Balance, December 31, 1951	9,884.77

	III. Carus Fund	IV. Chace Fund	V. Houck Fund	VI. Chauvenet Fund
Balance, January 1, 1951	$ 7,701.88	$10,117.07	$8,214.71	$626.05
Sale of Monographs	3,869.98	—	—	—
Sale of Papyrus	—	320.00	—	—
Sale of Slaught Papers	—	338.84	—	—
Sale of Index	—	—	14.60	—
Interest	272.00	355.69	292.92	20.92
Increase in value of securities	83.58	109.29	90.01	6.43
Balance, December 31, 1951	$11,927.44	$11,240.89	$8,612.24	$653.40

VII. General Fund

Balance, January 1, 1951	$33,292.37	Transfer to Current Fund	$ 4,484.95
Increase in value of securities	353.59	Balance, December 31, 1951	29,161.01

VIII. Total Funds of the Association on December 31, 1951

Current Fund	$ 9,884.77	M & T Trust Co., Buffalo	$ 9,884.77
Carus Fund	11,927.44	Securities	61,594.98
Chace Fund	11,240.89		
Houck Fund	8,612.24		
Chauvenet Fund	653.40		
General Fund	29,161.01		
	$71,479.75		$71,479.75

APRIL MEETING OF THE KANSAS SECTION

The thirty-sixth annual meeting of the Kansas Section of the Mathematical Association of America was held at the University of Kansas, Lawrence, Kansas, on Saturday, April 7, 1951. Professor L. M. Reagan, Chairman of the Section, presided. The morning session was a joint meeting with the Kansas Association of Teachers of Mathematics.

There were one hundred twenty-nine persons registered including the following fifty-nine members:

A. H. Albert, Don Alkire, P. H. Arnold, Wealthy Babcock, Florence L. Black, R. N. Bradt, J. H. Curtiss, Lucy T. Dougherty, Paul Eberhart, J. E. Faulkner, W. C. Foreman, J. W. Forman, Abraham Franck, Albert Furman, W. H. Garrett, F. C. German, W. L. Graves, Laura Z. Greene, J. R. Hanna, P. W. Healy, N. C. Hoover, D. M. Houser, K. C. Hsu, Emma Hyde, W. C. Janes, L. E. Laird, K. E. Lake, Violet H. Larney, C. F. Lewis, Anna Marm, Margaret E. Martinson, Thirza A. Mossman, J. D. Neff, Agnes E. Nibarger, S. T. Parker, O. J. Peterson, P. S. Pretz, G. B. Price, O. M. Rasmussen, C. B. Read, C. A. Reagan, L. M. Reagan, J. D. Riley, H. Q. Rolfs, R. G. Sanger, Robert Schatten, W. R. Scott, Sister M. Helen, G. W. Smith, R. G. Smith, E. C. Stopher, E. B. Stouffer, W. T. Stratton, Wilmont Toalson, C. B. Tucker, Frances E. Walsh, E. B. Wedel, Ferna E. Wrestler, P. M. Young.

The following officers were elected for the year 1951–1952: Chairman, Anna Marm, Bethany College; Vice-Chairman, P. M. Young, Kansas State College; Secretary-Treasurer, Laura Z. Greene, Washburn University.

The following papers were presented:

1. *Some novel methods for estimating the eigenvalues of linear operators*, by Dr. J. H. Curtiss, Chief, National Applied Mathematics Laboratories of the National Bureau of Standards.

The paper was concerned with probabilistic methods of computing the eigenvalues of certain linear operators. In particular, the time-independent Schrodinger equation $\frac{1}{2}\Delta u - Vu + \lambda u = 0$ was considered where V and the unknown function u are functions of several independent variables, Δ is the Laplacian operator, and λ is a constant. The pertinent boundary conditions are that u tends to zero as any coordinate becomes infinite, the squares of each of the first partial derivatives of u are each integrable over the entire plane, and the first partial derivatives have simple discontinuities at the origin. Two categories of classical methods of calculating the eigenvalues were first mentioned. These were: (1) the methods based on variational principles and (2) the methods based on substituting a difference equation for the differential equation, which reduces the problem to a matrix eigenvalue problem. The speaker then went on to describe a variant of the latter method developed by Dr. W. R. Wasow of the National Bureau of Standards, Los Angeles, in which the matrix eigenvalues are approximated by statistics obtained from random walks (see NBS Journal of Research, January, 1951).

2. *Algebraically closed groups*, by Professor W. R. Scott, University of Kansas.

Call a group G algebraically closed if, for every finite set S of equations and inequalities, either (i) there is a solution of S in G, or (ii) there is no solution of S in any group containing G as a subgroup. It is shown that every group G is a subgroup of an algebraically closed group H.

3. *Introducing the limit concept in elementary calculus*, by Professor P. W. Healy, Southwestern College.

It was pointed out that the examples used in the standard calculus textbooks do not serve to make clear to the beginning student the distinction between the value of a function at a point $x=a$ and its limit as $x\to a$. It was suggested that an intuitive approach be employed, using such functions as:

$$f(x)=\frac{2^x-1}{x} \quad \text{and} \quad f(x)=\frac{x^{\sqrt{2}}-1}{x-1}, \quad \text{and}$$

letting x approach zero and one respectively. Since these functions cannot be simplified to give them values at zero and one, the student is helped to see the importance of the limiting process. It is also advisable to introduce functions in which the value of the function and its limit are different at some particular point. This will further clarify the difference between the limit of a function and its value at a point.

4. *Introducing the undergraduate to the language of abstract algebra*, by Professor Violet H. Larney, Kansas State College.

The author stated her belief that every undergraduate major in mathematics should be given an opportunity to familiarize himself with some of the terms and basic concepts of abstract algebra. Not only would the prospective graduate student be more adequately prepared to begin his graduate studies, but also the student who does not plan to enter graduate school would gain a greater degree of mathematical maturity, a better understanding of algebra in general, and an awareness of some of the interesting topics in modern algebra. The approach can be through either a semester introductory course in abstract algebra, or through a renovated traditional course (such as theory of equations) into which is injected the modern algebra point of view. The present paper suggested a possible means for introducing each of the several topics to which the undergraduate should be exposed; the major terms under consideration were groups, fields, integral domains, and matrices.

5. *The spherical contour in precision tooling*, by Miss Agnes Nibarger, University of Wichita.

In making a contour on a tool to cut a sphere, the usual practice is to cut a groove in the tool with a grinding wheel whose periphery has the form of a torus. Since efficiency demands the cutting face to be a section of the groove taken at an angle less than a right angle, a distortion of the spherical surface results.

An improvement in cutting the contour by shaping the grinding wheel to a general torus is suggested. The general torus is defined as the surface formed by revolving a circle about a line not necessarily in its plane. The distance between the line and the plane of the circle is a variable depending upon the radius of the required sphere, the diameter of the grinding wheel, and the angle at which the cutting plane is taken across the groove.

6. *On the intersection of a straight line and an n-cell*, by Mr. Keith Moore, University of Kansas, introduced by the Secretary.

The points $x_0, x_1, \cdots, x_n$ of n-dimensional Euclidean space, $R^{(n)}$, will be called linearly independent if the vectors, (x_i-x_0), $(i=1, \cdots, n)$, are linearly independent. By the n-cell of $R^{(n)}$ with the linearly independent points $x_0, x_1, \cdots, x_n$ as vertices is meant the set of points $x=\sum_{i=0}^{n}a_ix_i$, where $\sum_{i=0}^{n}a_i=1$. The points for which $a_i>0$, for all i, can be shown to be interior points of the n-cell. The points for which at least one $a_i=0$ are boundary points. It is proved analytically that if x' is an interior point of an n-cell and if x'' is any other point of $R^{(n)}$, then the straight line through x' and x'' intersects the boundary of the n-cell in precisely two points.

LAURA Z. GREENE, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The thirtieth annual meeting of the Illinois Section of the Mathematical Association of America was held at the University of Illinois, Urbana, Illinois on Friday afternoon and Saturday forenoon, May 11 and 12, 1951. Professor W. T. McDaniel, Chairman of the Section, presided at all sessions.

There were ninety in attendance, including the following sixty-seven members of the Association:

H. G. Ayre, P. T. Bateman, D. R. Bey, A. H. Black, R. P. Boas, B. K. Brown, F. R. Brown, Josephine H. Chanler, B. B. Clark, D. R. Clutterham, J. J. Corliss, I. K. Feinstein, M. K. Fort, Jr., D. E. Freeland, A. E. Gault, N. A. Goldsmith, M. J. Gottlieb, R. T. Gregory, A. E. Hallerberg, M. C. Hartley, Corrine R. Hattan, L. L. Helms, F. E. Hohn, Dorothy S. Hooper, B. O. Hoyle, P. W. Ketchum, E. C. Kiefer, G. D. Kyle, L. L. Layton, Harry Levy, C. C. MacDuffee, C. T. McCormick, W. C. McDaniel, A. W. McGaughey, B. E. Meserve, H. J. Miles, E. B. Miller, C. N. Mills, Josephine M. Mitchell, G. E. Moore, M. G. Moore, Edna M. Norskog, F. S. Nowlan, Rufus Oldenburger, Daniel Orloff, J. W. Peters, W. L. Pickard, R. E. Pingry, E. W. Ploenges, Irma Reiner, Irving Reiner, T. E. Rine, L. A. Ringenburg, L. D. Rodabaugh, E. J. Schweppe, M. Anice Seybold, H. A. Simmons, Norma K. Stelford, R. C. Stephens, C. J. Stowell, E. H. Taylor, R. J. Teed, B. R. Ullsvik, L. R. VanDeventer, H. E. Vaughn, Alice K. Wright, R. K. Zeigler.

At the business meeting on Saturday morning the following officers were elected for the coming year: Chairman, S. S. Cairns, University of Illinois, Urbana, Illinois; Vice-Chairman, E. W. Schreiber, Western State College, Macomb, Illinois; Secretary-Treasurer, E. C. Kiefer, Millikin University, Decatur, Illinois. The annual meeting for 1952 will be held at Western State College, Macomb, Illinois, on Friday and Saturday, May 9–10, 1952.

The following program was presented:

1. *A remark on approximations to the length of the perimeter of an ellipse*, by Professor L. A. Ringenberg, Eastern Illinois State College.

This remark was suggested by a problem which appeared in the 1950 William Lowell Putnam Competition: *Two obvious approximations to the length L of the perimeter of the ellipse with semiaxes a and b are* $L_1=\pi(a+b)$ *and* $L_2=2\pi\sqrt{ab}$. *Which one comes nearer the truth when the ratio* b/a *is very close to* 1? A third approximation is $L_3=2\pi\sqrt{a^2+b^2/2}$. Using series expansions it is shown that $L_3>L>L_1>L_2$ and $L_3-L>L-L_1$ for any eccentricity.

2. *Inequalities*, by Professor R. P. Boas, Jr., Northwestern University.

The author discussed a group of inequalities for the coefficients of positive trigonometric polynomials and trigonometric series. He indicated how such problems arise in the study of crystal structure.

3. *The mathematics in Aristotle*, by Sister Marie Stephen, Rosary College, introduced by Sister Mariola, Rosary College.

This topic is interesting historically, for in it we find many of the concepts of irrationals, the continuum and infinite, debated and settled in terms of definitions prevalent in the day. The topic of the indivisible line is discussed at length to illustrate the genius of the philosopher in inspiring his students to pursue to the end his unfinished hints. Theophrastus proposes the arguments of the proponents of the indivisible line and then systematically refutes them in terms more philosophic than mathematical, showing that most of them either lead to absurdities or collide with estab-

lished mathematical theory including the concept of the continuous and the system of irrationals and incommensurables.

4. *A modified Dedekind cut technique for constructing the reals from the rationals*, by Professor L. D. Rodabaugh, Southern Illinois University.

In this paper the complete ordered field of real numbers is constructed from the minimal ordered field of rational numbers by means of a modified Dedekind cut technique. Each of the definitions of elements, relations and operations and of the constructions of identity and inverse elements is given in a single sentence. The immediate result is the construction of the real field at one stroke, so to speak, without the necessity of resorting either to footnoted amendments to the definitions or to the procedure of first constructing the positive reals and later extending to all reals by means of a number pair device.

5. *Defeatism in education*, by Professor C. C. MacDuffee, University of Wisconsin.

6. *The colleges and the national emergency*, by Professor E. W. Ploenges, Millikin University.

Present and pending legislation relative to the drafting of college students was discussed. No conclusions could be drawn, but several observations were made regarding male registration in the colleges next fall.

7. *The problem of the poorly prepared student* (*Panel Discussion*), by Professor H. A. Simmons, Northwestern University, Professor H. G. Ayre, Western Illinois State College, Professor M. C. Hartley, University of Illinois, Navy Pier, Chicago, and Professor M. Anice Seybold, North Central College.

Chairman McDaniel called for remarks from several members before the panel got under way, as an introduction to the general topic. Then Professor Simmons, acting as panel chairman, introduced the panel. Sometime before these meetings Professor Simmons had sent a questionnaire to some twenty colleges regarding their experience with the preparation of students to pursue college mathematics. He read the answers from several and commented on them, then made a few remarks about students coming to Northwestern University.

Professor Ayre then took the floor and discussed the subject as follows:

There is a preponderence of evidence that the average college freshman is poorly prepared in mathematics. This does not necessarily mean that the student is dull or has a low I.Q. He may only be the victim of circumstances. Secondary education has reached a new high as to confusion of objectives. In many cases the high school diploma is little more than a belated birth certificate or perhaps a certification of a more or less satisfactory record of attendance at school for twelve years. The colleges have little choice in the matter of accepting the high school product and must start the students where they are and proceed. It is recommended that tests of mathematical competence be administered to incoming freshmen and the course adjusted accordingly. At first there should be a very careful presentation of elementary topics from a new viewpoint, if possible. The beginning freshman needs review, he needs encouragement, and he surely needs to establish confidence in his ability to master mathematics. Time spent in orienting the student and creating a will to learn will pay big dividends later. Many students may need to spend as much as sixteen to twenty quarter hours on pre-calculus mathematics. The speaker recommends that college credit be given for the work on elementary topics; however, it might be well to limit credit towards a mathematics major or minor to approximately sixteen quarter hours for work in pre-calculus mathematics.

Then Professor Hartley read a lengthy paper, the abstract of which follows. The problem of the poorly prepared student can be approached directly and indirectly. The direct approach can be broken down into five attacks. First the administration should establish for each freshman

course rigid prerequisites in terms of present ability. The administration should assign its best teachers to the classes populated by poorly prepared students. The pedagogical approach is the approach of the good teacher—the teacher who studies his students, adapts his instruction to their characteristics and is willing to pioneer in untried procedures. The scientific approach has two important aspects. First we need to know how to teach so that a process which has been learned incorrectly can be completely eliminated and then learned correctly. Second we need a complete reevaluation of our teaching methods. Common errors indicate a wide "learning gap" between the first presentation of a topic and the actual recitation a year or more later. The explication will be found in analyzing and experimenting with each phase of every problem of instruction. The poorly prepared student has no desire to make his knowledge as complete as possible, he likes his own methods, he does not accept the judgment of his teacher, he feels no need to study because previously he has been passed for good behavior and regular attendance. The psychological approach must deal with the breaking down and elimination of these poor attitudes and their associated habits, and with the building up of desirable ones. The curricular approach includes the establishment of courses adapted to the ability of the student, and might well include a series of short courses of not more than four weeks' duration.

The indirect approach to the problem can be broken down into two attacks: pressures can be brought by the accrediting agencies upon the secondary school so that a high school diploma will be a symbol of competence in the subjects pursued, and a guarantee that the minimum essentials of each course have been mastered. At the other extreme, parents can bring pressure upon the schools to turn out a better product. Students are not at fault. The system in which they are entangled is to blame!

Professor Seybold was the last speaker on the panel and closed with the following remarks: She reported on the mathematics preparation of freshmen who enter North Central College, stating that in the last three years the percentage of freshmen offering three or more years of high school mathematics has dropped from 46 to 40. For the students who are not properly prepared for the courses in which they are registered an optional laboratory hour is offered each week. According to a survey of twenty-one catalogs of colleges mostly in the state of Illinois, fourteen of these colleges were offering high school mathematics courses with or without college credit. The most commonly offered high school mathematics courses were intermediate algebra and solid geometry. It was urged that mathematicians take action in reporting to educators their stand on high school mathematics preparation.

E. C. KIEFER, *Secretary*

THE MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The annual meeting of the Upper New York State Section of the Mathematical Association of America was held at Hamilton College, Clinton, New York, on Saturday, May 5, 1951. The Chairman of the Section, President B. C. Patterson of Washington and Jefferson College, presided at the morning session; the Vice-Chairman, Professor C. W. Munshower of Colgate University, presided at the afternoon session. At the conclusion of the afternoon session tea was served.

Ninety persons attended the meeting, including the following seventy-five members of the Association:

E. B. Allen, H. P. Atkins, H. T. R. Aude, Frederick Bagemihl, D. C. Barton, M. R. Bates, R. W. Beals, Jr., R. A. Beaver, W. R. Beck, R. L. Beinert, Harry Birchenough, F. J. H. Burkett, E. A. Butler, J. D. Campbell, R. C. Clelland, Nancy Cole, E. J. Downie, Jean B. Feidner, C. W. Foard, A. H. Fox, J. E. Freund, W. H. Fuchs, H. M. Gehman, B. H. Gere, Lillian Gough, N. G. Gunderson, R. G. Hill, H. K. Holt, Anna M. Howe, R. E. Huston, J. R. F. Kent, D. E. Kibbey,

R. D. Larsson, Caroline A. Lester, J. V. Limpert, Charles Loewner, R. C. Luippold, A. W. Mall, Dis Maly, June M. McArtney, L. L. Merrill, Norman Miller, Harriet F. Montague, Mabel D. Montgomery, D. S. Morse, C. W. Munshower, E. E. Nash, W. V. Nevins, C. V. Newsom, W. N. Palmer, B. C. Patterson, Ruth M. Peters, Theresa L. Podmele, L. R. Polan, J. F. Randolph, C. E. Rhodes, P. C. Rosenbloom, M. F. Rosskopf, G. X. Saltarelli, Edith R. Schneckenburger, Wladimir Seidel, Esther Seiden, R. E. Shear, William A. Smith, M. R. Spiegel, Mary R. Steudle, Ruth W. Stokes, Mary C. Suffa, Nura D. Turner, J. D. van Alstyne, G. W. Walker, A. K. Waltz, J. F. Wardwell, F. C. Warner, Frances M. Wright.

The following officers were elected: Chairman, C. W. Munshower, Colgate University; Vice-Chairman, J. F. Randolph, University of Rochester; Secretary, N. G. Gunderson, University of Rochester. The Chairman was authorized to appoint a committee to study the question of sponsoring a mathematics contest for high school students. The 1952 meeting will be held at Hobart and William Smith Colleges, Geneva, New York, in May.

The following papers were presented:

1. *A doubly oscillating pendulum*, by Professor A. K. Waltz, Clarkson College of Technology.

The speaker considered the following problem: Replace the rod of a simple pendulum by a coiled spring of negligible weight. Let the spring swing under gravity as a simple pendulum, friction neglected, and at the same time oscillate longitudinally. If initially $\rho=l_0$, $\theta=\theta_0$, $(\dot\theta)^2=(\dot\theta)_0^2$, and $(\dot\rho)^2=(\dot\rho)_0^2$, find the polar equation of the centroid of the pendulum bob.

Resolving forces along and perpendicular to the radius vector (with notation as for simple pendulum):

(1) $$\ddot\rho - \rho(\dot\theta)^2 = g\cos\theta - k(\rho-\rho_0), \quad \rho_0 = \text{natural spring length},$$

(2) $$\frac{1}{\rho}\frac{d(\rho^2\dot\theta)}{dt} = -g\sin\theta, \qquad k = \text{spring constant}, m = 1.$$

When $\dot\theta$ times (2) is subtracted from the time derivative of (1), then 2ρ becomes an integrating factor yielding

(3) $$2\rho\dot\rho\ddot\rho - (\dot\rho)^2 - 3(\rho\dot\rho)^2 + k\rho^2 = C.$$

The velocity function

(4) $$v^2 = (\rho\dot\theta)^2 + (\dot\rho)^2 = 2g\rho\cos\theta - k\rho^2 + 2k\rho_0\rho - C$$

is then found by eliminating $\ddot\rho$ from (1) and (3).

Now since equation (2) may be written

(2b) $$(\rho\dot\theta)^2 = -2g\int\rho^2\sin\theta d\theta$$

the following equation in ρ and θ is found by eliminating $(\dot\theta)^2$ from (4) and (2b) with the aid of $(\dot\rho)^2=(d\rho/d\theta)^2(\dot\theta)^2$:

(5) $$\rho^4(2g\rho\cos\theta - 2k\rho^2 + 2k\rho_0\rho - C) + 2g\left[\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2\right]\int\rho^2\sin\theta d\theta = 0.$$

This equation (5) together with initial conditions determine the coefficients in an assumed solution of the form $\cos\theta=\sum_{n=0}^{\infty}a_n\rho^n$. The resulting solution is valid except when $k\to\infty$. Then $\dot\rho\equiv 0$, $\rho=\rho_0$, equation (1) becomes nil, and the solution reduces to (2) with $\rho=\rho_0$, giving rise to the simple pendulum.

Each a_n (except a_0) contains the factor a_0^2-1. Hence when $a_0^2-1=0$, $(\dot{\theta})^2\equiv 0$; $\cos\theta\equiv 1$, equation (2) becomes nil, and the solution reduces to (1) with these values inserted giving the special case of a coiled spring vibrating vertically.

2. *The bivariate binomial distribution and applications*, by Professor J. E. Freund, Alfred University.

Professor Freund developed the distribution function of the bivariate binomial distribution as well as its moment generating function and then applied his results to problems in correlation and sampling inspection. Generalizing the discussion to the multivariate case, the speaker showed how the fourvariate binomial distribution can be employed as a convenient model to evaluate the effect of the "human element," *i.e.*, the bias which is introduced by the "observer," in the estimation of probabilities. This mathematical model was used, in particular, as the basis of a reply to criticisms raised by Bertrand Russell against the statistical concept of probability.

3. *Extremal problems treated by elementary methods*, by Professor W. H. Fuchs, Cornell University.

The speaker made a plea for the inclusion of elementary geometrical extremal problems in the classroom treatment of maxima and minima. He discussed several examples, in particular the rectangle of largest area circumscribed to a given quadrangle, and the least ring through which a regular tetrahedron can be slipped.

4. *Report of the committee on the relations of secondary school and college mathematics*, Professor M. F. Rosskopf, Syracuse University.

On the basis of its deliberations, the committee recommended that the Upper New York State Section (1) encourage the formation of secondary school honor societies in mathematics, (2) promote mathematics clubs in which both secondary school and college mathematics teachers would be members, (3) conduct mathematics contests for high school students. The concrete results of committee work are (1) appointment of a committee to supervise and to plan a mathematics contest in Buffalo, New York, to be held in the spring of 1952; (2) active participation in the formation of a state-wide organization of mathematics teachers, the Association of Mathematics Teachers of New York State.

5. *A theorem in measure theory and applications to statistics*, by Professor Esther Seiden, University of Buffalo.

This speaker gave a discussion on the existence of an unbiased test for testing a composite statistical hypothesis against a composite alternative. Conditions under which such a test exists were stated.

6. *Some classes of power series*, by Professor Frederick Bagemihl, University of Rochester.

Let K be the class of complex power series having the unit circle as circle of convergence. Denote by C the class of those elements of K which can be continued analytically beyond the unit circle, and by A the class of those elements of K which satisfy an algebraic differential equation, and let C', A' be the complements of C, A with respect to K. Then each of the classes CA, $C'A$, CA', $C'A'$ has the power of the continuum. Any element of K can be transformed into an element of $C'A'$ by merely multiplying certain terms by -1, and this can be done in continuum many ways. (This generalizes results of Polya and Ostrowski.) There exist continuum many elements of $C'A'$, each having all its coefficients rational and different from zero, such that, no matter which terms are multiplied by -1, the resulting series all belong to $C'A'$.

7. *Some inequalities in differential geometry*, by Professor Charles Loewner, Syracuse University.

Consider in the x, y plane a Jordan domain D and mark on its boundary curve B four points A_1, A_2, A_3, A_4 in their natural order. One might call D a pseudo-rectangle with the vertices A_1, A_2, A_3, A_4. Consider further all curves in D connecting the "opposite sides" of the pseudo-rectangle formed by the arcs A_1A_2 and A_3A_4 on B and call the greatest lower bound of their lengths a. Introduce finally a similar quantity b by using the other pair of opposite sides A_2A_3 and A_1A_4. The quantities a and b may be called the sides of the pseudorectangle B. The following inequality holds

$$A \geqq ab \tag{1}$$

with A representing the area of D. The equality sign holds for a euclidean rectangle. The establishment of this inequality gives rise to questions of the following general character: On a Riemannian, or more generally a Finslerian manifold M of dimension 2, one or several classes of curves $C_1, C_2, \cdots, C_n$ are given and in each class the lengths of the curves have a positive greatest lower bound $a_1, a_2, \cdots, a_n$ respectively. Can a lower bound for the area A of M depending only on the a_i be established? The following results were obtained: (2) The inequality (1) holds for an arbitrary Riemannian metric; (3) If more generally a Finsler metric with a convex symmetric indicatrix is used and the area is suitably defined, (1) has to be replaced by

$$A \geqq \frac{2}{\pi} ab.$$

Of particular interest is the case where only one class C of curves is considered, namely the totality of closed curves which cannot be contracted to a point. The following results were obtained: (4) If the underlying topological space is a torus then $A \geqq a^2\sqrt{3}/2$; (5) If the space is topologically a projective plane then $A \geqq 2a^2/\pi$. The last result is contained in the doctoral thesis of Mr. P. M. Pu.

8. *The sum of a generalized harmonic series and some related results*, by Professor M. R. Spiegel., Rensselaer Polytechnic Institute.

A physical problem arising in potential theory led to the mathematical one of finding the sum of a "generalized harmonic series" $1/a-1/(a+b)+1/(a+2b)-\cdots$ where a and b are integers such that $1 \leqq a \leqq b$. This sum is found to be equal to

$$\frac{\pi}{2b \sin(\pi a/b)} - \frac{2}{b} \sum_{k=1}^{[b/2]} \left\{ \cos (2k-1) \frac{\pi a}{b} \right\} \left\{ \log_e \sin (2k-1) \frac{\pi}{2b} \right\}$$

where $[x]$ denotes, as usual, the largest integer not greater than x. Thus, for example, if $a=1$, $b=6$, we have

$$1 - \frac{1}{7} + \frac{1}{13} - \cdots = \frac{\pi}{6} + \frac{\sqrt{3}}{6} \log_e (2+\sqrt{3}).$$

The generalized harmonic series can also be expressed in terms of the logarithmic derivative of the Γ function, and one may derive the expression

$$\Gamma'\left(\frac{a}{b}\right) \Big/ \Gamma\left(\frac{a}{b}\right) = 2 \left\{ \sum_{k=1}^{[b/2]} \left(\cos \frac{2k\pi a}{b} \right) \left(\log_e \sin \frac{k\pi}{b} \right) \right\} - \log_e 2b - \frac{\pi}{2} \cot \frac{\pi a}{b} - \gamma$$

where γ is Euler's constant and the other symbols have the same significance as before. From these results, many integrals, some of which arise in analytic number theory, may be evaluated in closed form.

N. G. GUNDERSON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Summer Meeting, Michigan State College, East Lansing, Michigan, September 1–2, 1952.

Thirty-sixth Annual Meeting, Washington University, St. Louis, Missouri, December 30, 1952.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Waynesburg College, Waynesburg, Pennsylvania, May 10, 1952.
ILLINOIS, Western Illinois State College, Macomb, May 9–10, 1952.
INDIANA, Indiana University, Bloomington, May 3, 1952.
IOWA, Coe College, Cedar Rapids, April 18–19, 1952.
KANSAS
KENTUCKY, University of Kentucky, Lexington, April 19, 1952.
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Virginia Military Institute, Lexington, April 26, 1952.
METROPOLITAN NEW YORK, Hofstra College, Hempstead, New York, May 3, 1952.
MICHIGAN, University of Michigan, Ann Arbor, April 12, 1952.
MINNESOTA, College of St. Catherine, St. Paul, May 10, 1952.
MISSOURI, Lindenwood College, St. Charles, May 2, 1952.
NEBRASKA, University of Nebraska, Lincoln, May 3, 1952.
NORTHERN CALIFORNIA
OHIO, Ohio State University, Columbus, April 19, 1952.
OKLAHOMA
PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.
PHILADELPHIA, University of Delaware, Newark, November 29, 1952.
ROCKY MOUNTAIN, Western State College, Gunnison, Colorado, May 23–24, 1952.
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN, University of Arizona, Tucson, April 11–12, 1952.
TEXAS, East Texas State Teachers College, Commerce, April 25–26, 1952.
UPPER NEW YORK STATE, Hobart and William Smith Colleges, Geneva, May 10, 1952.
WISCONSIN, Milwaukee, May 10, 1952.

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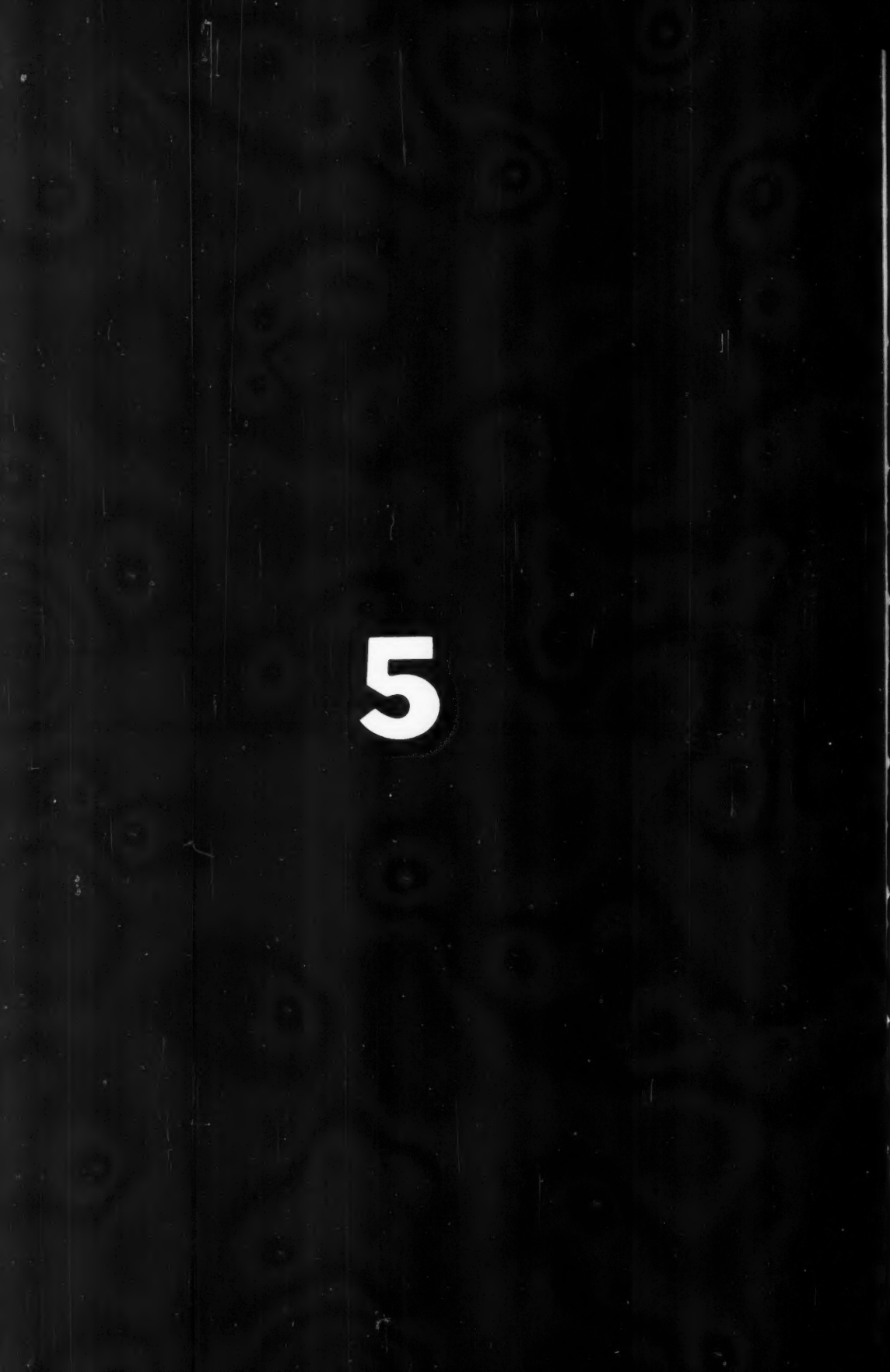
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